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Computer Simulation of Einstein-Podolsky-Rosen-Bohm Experiments

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Abstract. We review an event-based simulation approach which reproduces the statistical distributions of quantum physics experiments by generating detection events one-by-one according to an unknown distribution and without solving a wave equation. Einstein-Podolsky-Rosen-Bohm laboratory experiments are used as an example to illustrate the applicability of this approach. It is shown that computer experiments that employ the same post-selection procedure as the one used in laboratory experiments produce data that is in excellent agreement with quantum theory.

1. Introduction

Computer simulation is widely regarded as complementary to theory and experiment [1]. Traditionally, in applications to physics, one starts from one or more of the basic equations of physics and uses the computer as a calculator to carry out arithmetics according to an algorithm derived from or specified by these basic equations.

Computer experiments such as the ones discussed in this paper are conceptually very different from man-made mathematical theories. For instance, in the latter, the concept of a real number such as π plays a central role. Digital computers, which necessarily have finite resources, work with integer and floating-point numbers that, strictly speaking, do not obey the rules of standard arithmetic. Of course, a digital computer can be used to manipulate symbols that are encoded as bits, symbols that represent mathematical constructs such as real numbers but that is not the point. A digital computer is just a physical device, the state (i.e. the collection of all its bits) of which changes according to certain rules. Running an algorithm on a digital computer is nothing but carrying out a real experiment for which there are no unknown or uncertain factors that might affect the outcome. Therefore the digital computer + the rules can be viewed as a metaphor for a laboratory experiment.

This metaphor can be carried further. As is well-known, a digital computer can be realized not only with the semiconductor technology that we are all familiar with these days but also by purely mechanical means, e.g. a Babbage programmable machine. The use of the latter metaphor has an important conceptual consequence. Indeed, if an algorithm running on a digital computer reproduces the results of a theory without making reference to the basic equation(s) of that theory, it has been established that, at least for this particular instance, there exists a (macroscopic) mechanical device that produces the same data. In other words, there is no need to invoke concepts of the said theory to explain the facts, at least not for this particular case.

Applied to quantum physics experiments, this means the following. In the traditional approach, we would typically solve the Schrödinger equation to obtain a probability distribution. The individual events, ultimately registered by our senses, are then generated by means of pseudo-random numbers drawn from this probability distribution. An alternative approach is to invent algorithms that generate the same type of events as observed in experiment such that, for instance, interference patterns appear only after a considerable number of individual events have been recorded by the detector [2, 3, 4], and all this without solving the Schrödinger equation, of course.

The central question addressed by such alternative approach can be posed as follows: Can we construct an algorithm which runs on a digital computer and produces events with frequencies that agree with those observed in laboratory experiments without the algorithm referring, in any way, to the probability distribution that is predicted by theory? An affirmative answer to this question implies, through the computer-experiment metaphor, that we have found a set of rules by which the digital computer acts as an experiment, producing data that is of the same type and has similar frequency distributions as the data that is generated in the laboratory experiment.

The basic ideas of the discrete-event computer simulation approach that we review in this paper are that (i) we stick to what we know about the experiment, that is we consider the experimental configuration, its outcome and its data analysis procedure as input for constructing the simulation algorithm; (ii) we try to invent a set of simple rules that generates the same type of data as those recorded in an experiment while reproducing the averages predicted by quantum theory; (iii) we keep compatibility with macroscopic concepts, which in view of the experiment-computer-macroscopic mechanical device is not a surprise.

The discrete-event simulation methodology has successfully been applied to all of the most fundamental quantum physics experiments performed with neutrons as well as photons that involve interference, uncertainty, and entanglement [5 – 9]. In this paper, we review its application to Einstein-Podolsky-Rosen-Bohm laboratory experiments and the related subject of violations of Bell-type inequalities. From the viewpoint of computer simulation, this application is the simplest, nontrivial case to consider. Applications that involve interference require rules that are more sophisticated but backwards compatible with the ones that suffice to simulate Einstein-Podolsky-Rosen-Bohm laboratory experiments [10, 6, 7, 8].

2. Einstein-Podolsky-Rosen-Bohm Experiments

In 1935, Einstein, Podolsky and Rosen (EPR) proposed a thought experiment to address the question of incompleteness of quantum theory [12]. The thought experiment involves the measurement of the position and momentum of two particles which interacted in the past but not at the time of measurement. This thought experiment is difficult to perform in a laboratory.

In 1951 Bohm proposed another experiment [11] which addresses the same question and has the potential to be realized in a real laboratory, a point which is discussed further in Sect. 4. A schematic diagram of Bohm's version of the EPR thought experiment, which will be called Einstein-Podolsky-Rosen-Bohm (EPRB) thought experiment in the following, is shown in Fig. 1. A source emits charge-neutral pairs of particles with opposite magnetic moments $+\boldsymbol{S}$ and $-\boldsymbol{S}$. The two particles separate spatially and propagate in free space to an observation station in which they are detected. As the particle arrives at station $i = 1, 2$, it passes through a Stern-Gerlach magnet. The magnetic moment of a particle interacts there with the inhomogeneous magnetic field. The Stern-Gerlach magnet deflects the particle, depending on the orientation of the magnet $\boldsymbol{a}_i$ and the magnetic moment of the particle and it divides the beam of particles into two spatially well-separated parts. As the particle leaves the magnet, it generates a signal $x_i = \pm 1$ in one of the two detectors $\mathrm{D}_{\pm,i}$. The firing of a detector corresponds to the detection event. The quantity of interest is the correlation of the intrinsic angular momentum (magnetic moments, spins) of the particles.

In the early 1960's, Bell derived his famous inequalities. Originally believed by Bell to be a vehicle to prove quantum theory wrong [71], these inequalities were later thought to be useful for establishing the existence of an action on a distance [70, 46]. Many researchers questioned the relevance of Bell's inequalities for addressing such issues [13 – 46] but his work had a major impact in that it spurred many experimental efforts to implement the (vari-

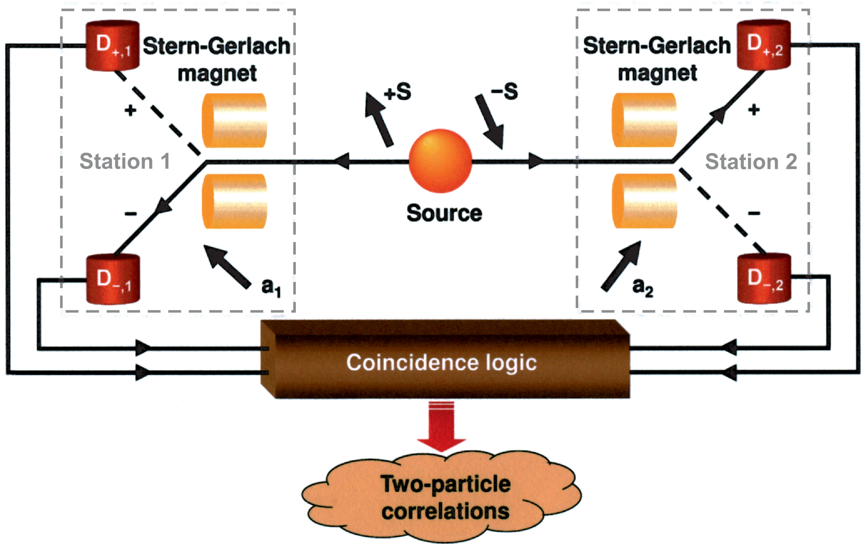


Fig. 1: (Color online) Schematic diagram of the Einstein-Podolsky-Rosen-Bohm (EPRB) experiment with magnetic particles [11]. The source emits charge-neutral pairs of particles with opposite magnetic moments $+\boldsymbol{S}$ and $-\boldsymbol{S}$. One of the particles moves to station 1 and the other one to station 2. As the particle arrives at station $i = 1, 2$, it passes through a Stern-Gerlach magnet which deflects the particle, depending on the orientation of the magnet $\boldsymbol{a}_i$ and the magnetic moment of the particle. As the particle leaves the Stern-Gerlach magnet, it generates a signal in one of the two detectors $\mathrm{D}_{\pm,i}$. Coincidence logic pairs the detection events of station 1 and station 2 so that they can be used to compute two-particle correlations.

ations of) EPRB thought experiment. As a result, in scientific and popular literature on quantum physics, it is quite common to find statements that the experimental results [47–59] of EPRB experiments confirm the predictions of quantum theory.

While it is firmly established that the experimental data indeed do show violations of a Bell inequality, it is remarkable that this observation alone is taken to imply that the experimental data gathered in EPRB experiments does indeed comply with the predictions of quantum theory for this particular experiment. However, the most elementary analysis of the data gathered in three very different experiments [53, 60, 61, 62] show the same feature, namely that the statistical variation of data produced by these EPRB experiments is much larger than for most experiments which made quantum theory famous [63, 64]. For instance, the energy of photons emitted from atoms is reproducible to many digits of accuracy so that the theory and experiment agree very well but data of EPRB experiments deviate significantly (more than 4–5 standard deviations) from quantum theoretical predictions [63, 64]. This finding does not prove quantum theory wrong: it merely indicates that the experimental data are not described by quantum theory of the EPRB thought experiment (e.g. this model is too simple) or that more precise experiments are called for.

In terms of controllability, experiments with photons are no match for computer experiments and are therefore much harder to perform. Moreover, it has been shown in [65], without taking recourse to quantum theory, that under very general conditions, the statistics of a close-to-perfect, robust EPRB experiment is determined by the probability

$$P(x_1, x_2|\boldsymbol{a}_1, \boldsymbol{a}_2) \;=\; \frac{1}{4}(1 - x_1 x_2 \boldsymbol{a}_1 \cdot \boldsymbol{a}_2)$$

(i.e. the probability distribution of a singlet state), hence the success of such an endeavor is guaranteed.

3. Quantum-Theoretical Description

3.1. Single- and two-particle averages

According to the quantum theoretical description of the EPRB thought experiment, the results of repeated measurements of the system of two spin-1/2 particles in the quantum state $|\Psi\rangle = c_0\,|\uparrow\uparrow\rangle + c_1\,|\downarrow\uparrow\rangle + c_2\,|\uparrow\downarrow\rangle + c_3\,|\downarrow\downarrow\rangle$ with $\sum_{j=0}^{3}|c_j|^2 = 1$ are given by the single-spin and two-spin expectation values

$$\begin{aligned} \widehat{E}_1(\boldsymbol{a}_1) &= \langle\Psi|\boldsymbol{\sigma}_1\cdot\boldsymbol{a}_1|\Psi\rangle = \langle\Psi|\boldsymbol{\sigma}_1|\Psi\rangle\cdot\boldsymbol{a}_1,\\ \widehat{E}_2(\boldsymbol{a}_2) &= \langle\Psi|\boldsymbol{\sigma}_2\cdot\boldsymbol{a}_2|\Psi\rangle = \langle\Psi|\boldsymbol{\sigma}_2|\Psi\rangle\cdot\boldsymbol{a}_2,\\ \widehat{E}_{12}(\boldsymbol{a}_1,\boldsymbol{a}_2) &= \langle\Psi|\boldsymbol{\sigma}_1\cdot\boldsymbol{a}_1\boldsymbol{\sigma}_2\cdot\boldsymbol{a}_2|\Psi\rangle \;=\; \boldsymbol{a}_1\cdot\langle\Psi|\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\Psi\rangle\cdot\boldsymbol{a}_2, \end{aligned} \tag{1}$$

where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are unit vectors specifying the directions of the analyzers, $\boldsymbol{\sigma}_i$ denote the Pauli vectors describing the spin of the particles $j = 1, 2$. We have introduced the notation $\widehat{\ }$ to make a distinction between quantum theoretical results and the results obtained from simulation/experiments.

Quantum theory of the EPRB thought experiment further assumes that $|\Psi\rangle$ does not depend on $\boldsymbol{a}_1$ or $\boldsymbol{a}_2$. Therefore, from (1) it follows immediately that $\widehat{E}_1(\boldsymbol{a}_1)$ does not depend on $\boldsymbol{a}_2$ and that $\widehat{E}_2(\boldsymbol{a}_2)$ does not depend on $\boldsymbol{a}_1$. Note that this holds for any state $|\Psi\rangle$ or, more generally, for any state described by a density matrix which does not depend on $\boldsymbol{a}_1$ or $\boldsymbol{a}_2$.

The quantum theoretical description of the EPRB experiment assumes that the state of two spin-1/2 particles is described by the singlet state $\rho =$

$|\Psi\rangle\langle\Psi|$, where

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right). \tag{2}$$

For the singlet state, $\widehat{E}_1(\boldsymbol{a}_1) = \widehat{E}_2(\boldsymbol{a}_2) = 0$, $\widehat{E}_{12}(\boldsymbol{a}_1, \boldsymbol{a}_2) = -\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$, the correlation $\widehat{\rho}_{12}(\boldsymbol{a}_1, \boldsymbol{a}_2) = \widehat{E}_{12}(\boldsymbol{a}_1, \boldsymbol{a}_2) - \widehat{E}_1(\boldsymbol{a}_1)\widehat{E}_2(\boldsymbol{a}_2) = \widehat{E}_{12}(\boldsymbol{a}_1, \boldsymbol{a}_2)$. Note that the quantum state of two spins is fully characterized by the three quantities $\widehat{E}_1(\boldsymbol{a}_1)$, $\widehat{E}_2(\boldsymbol{a}_2) = 0$, and $\widehat{E}_{12}(\boldsymbol{a}_1, \boldsymbol{a}_2)$.

If two spin-1/2 particles are in a product state $|\Psi\rangle = |\Psi\rangle_1|\Psi\rangle_2$ with $|\Psi\rangle_j = c_{0,j}\,|\uparrow\rangle_j + c_{1,j}\,|\uparrow\rangle_j$ with $|c_{0,j}|^2 + |c_{1,j}|^2 = 1$ for $j = 1, 2$, then $\widehat{E}_{12}(\boldsymbol{a}_1, \boldsymbol{a}_2) = \langle\Psi|\boldsymbol{\sigma}_1|\Psi\rangle_1 \cdot \boldsymbol{a}_1\langle\Psi|\boldsymbol{\sigma}_2|\Psi\rangle_2 \cdot \boldsymbol{a}_2$, hence the correlation $\widehat{\rho}_{12} = \widehat{E}_{12}(\boldsymbol{a}_1, \boldsymbol{a}_2) - \widehat{E}_1(\boldsymbol{a}_1)\widehat{E}_2(\boldsymbol{a}_2) = 0$.

Many implementations of the EPRB experiment use photons instead of massive particles. Then the polarization of each photon plays the role of the spin-1/2 degree-of-freedom. Using the fact that the two-dimensional vector space with basis vectors $\{|H\rangle, |V\rangle\}$, where H and V denote the horizontal and vertical polarization of photons, respectively, is isomorphic to the vector space with basis vectors $\{|\uparrow\rangle\,, |\downarrow\rangle\}$ of spin-1/2 particles, we may use the quantum theory of the latter to describe the experiments with photons. For photons, the singlet state reads

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1|V\rangle_2 - |V\rangle_1|H\rangle_2\right) = \frac{1}{\sqrt{2}}\left(|HV\rangle - |VH\rangle\right), \tag{3}$$

and the uncorrelated quantum state reads

$$|\Psi\rangle = \left(\cos\zeta_1|H\rangle_1 + \sin\zeta_1|V\rangle_1\right)\left(\cos\zeta_2|H\rangle_2 + \sin\zeta_2|V\rangle_2\right), \tag{4}$$

where ζ_j for $j = 1, 2$ denote the definite polarization of photons and the subscripts refer to photon 1 and 2, respectively. The polarization vector $\boldsymbol{P}_j = (\cos\zeta_j, \sin\zeta_j, 0)$ replaces the magnetic moment

$$\boldsymbol{S}_j = (\cos\phi_j \sin\theta_j, \sin\phi_j \sin\theta_j, \cos\theta_j)$$

of the spin-1/2 particle.

The expressions for the single-photon expectation values and the two-photon correlations are similar to those of the genuine spin-1/2 particle problem except for the restriction of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ to lie in planes orthogonal to the direction of propagation of the photons and that the polarization is defined modulo π, not modulo 2π as it is in the case of the spin-1/2 particles. The latter results in a multiplication of the angles by a factor of two. For simplicity it is often assumed that $\boldsymbol{a}_j = (\cos a_j, \sin a_j, 0)$ for $j = 1, 2$. The resulting single and two-particle expectation values and the correlations are summarized in Table 1.

Table 1: Single and two-particle expectation values for a quantum system of two photons in the singlet state and the uncorrelated quantum state.

	Singlet state	Uncorrelated quantum state
$\widehat{E}_1(a_1)$	0	$\cos 2(\zeta_1 - a_1)$
$\widehat{E}_2(a_2)$	0	$\cos 2(\zeta_2 - a_2)$
$\widehat{E}(a_1, a_2)$	$-\cos 2(a_1 - a_2)$	$\cos 2(\zeta_1 - a_1) \cos 2(\zeta_2 - a_2)$
$\widehat{\rho}_{12}(a_1, a_2)$	$-\cos 2(a_1 - a_2)$	0

In any laboratory, thought or computer experiment which aims at demonstrating that the system may be described by a singlet state, it is absolutely necessary to show that the results for the three quantities $\widehat{E}_1(\boldsymbol{a}_1)$, $\widehat{E}_2(\boldsymbol{a}_2) = 0$ and $\widehat{E}(\boldsymbol{a}_1, \boldsymbol{a}_2)$, and not only for the latter (which seems to be the standard in all laboratory experiments reported upto now) comply with the predictions of quantum theory.

3.2. Bell-type inequalities

From the algebraic identity $(1 \pm xy)^2 = (x \pm y)^2 + (1 - x^2)(1 - y^2)$ it follows that $|x \pm y| \le 1 \pm xy$ for real numbers $|x| \le 1$ and $|y| \le 1$. From this inequality it immediately follows that

$$|xz \pm yz| \ \le \ 1 \pm xy \,, \tag{5}$$

for real numbers x, y, z such that $|x| \le 1$, $|y| \le 1$ and $|z| \le 1$. From (5) it follows immediately that

$$|xz - yz + xz' + yz'| \ \le \ 2 \,, \tag{6}$$

if, in addition, z' is a real number such that $|z'| \le 1$.

If the two spin-1/2 particles are described by a product state, we can use (5) and with unit vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$ obtain a Bell-type inequality

$$|\widehat{E}(\boldsymbol{a}_1, \boldsymbol{a}_2) - \widehat{E}(\boldsymbol{a}_1, \boldsymbol{a}_3)| \ \le \ 1 + \widehat{E}(\boldsymbol{a}_2, \boldsymbol{a}_3) \,, \tag{7}$$

and the Bell-CHSH (Clauser-Holt-Shimony-Horne) inequality [66]

$$-2 \ \le \ \widehat{E}(\boldsymbol{a}_1, \boldsymbol{a}_3) - \widehat{E}(\boldsymbol{a}_1, \boldsymbol{a}_4) + \widehat{E}(\boldsymbol{a}_2, \boldsymbol{a}_3) + \widehat{E}(\boldsymbol{a}_2, \boldsymbol{a}_4) \ \le \ 2 \,, \tag{8}$$

where, for later use, it is expedient to introduce the function

$$\widehat{S} \ = \ \widehat{S}(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4) \ = \ \widehat{E}(\boldsymbol{a}_1, \boldsymbol{a}_3) - \widehat{E}(\boldsymbol{a}_1, \boldsymbol{a}_4) + \widehat{E}(\boldsymbol{a}_2, \boldsymbol{a}_3) + \widehat{E}(\boldsymbol{a}_2, \boldsymbol{a}_4) \,. \tag{9}$$

Within the quantum theory of two spin-1/2 particles, it can be shown that $|\widehat{S}| \leq 2\sqrt{2}$, independent of the choice of the quantum state [67]. If the quantum system is described by a product (singlet) state, we necessarily have $|\widehat{S}| \leq 2$ ($|\widehat{S}| = 2\sqrt{2}$).

From the conditions under which (7) and (8) have been derived it follows immediately that if the state of the two spin-1/2 particle system is a product state, then both Bell and Bell-CHSH inequalities hold.

On the other hand, if the Bell or Bell-CHSH inequality is violated, then the two-particle quantum system cannot be described by a product state. Note that these logical statements are made entirely within the framework of quantum theory and that, within this context, these are the only logically valid conclusions that can be drawn from a (non)violation of the Bell or Bell-CHSH inequality [45]. For discussions of Bell-inequalities in the context of (non) Kolmogorovian probability theory see [43, 68] and references therein.

4. Computer Simulation of EPRB Experiments with Photons

4.1. Laboratory experiments

We take the EPRB experiment with single photons, carried out by Weihs *et al.* [53, 54], as a concrete example. We first describe data collection and the analysis procedure of the experiment. Then we illustrate how to construct an event-based model of an idealized version of this experiment which reproduces the predictions of quantum theory for the single and two-particle averages for a quantum system of two spin-1/2 particles in the singlet state without making reference to the formalism of quantum theory.

Data collection

Figure 2 shows a schematic diagram of the EPRB experiment with single photons carried out by Weihs *et al.* [53, 54]. The source emits pairs of photons with orthogonal but otherwise random polarization. The photon pair splits and each photon travels in free space to an observation station, labelled by $j = 1$ or $j = 2$, where it is manipulated and detected. The two stations are assumed to be identical and are separated spatially and temporally. Hence, the observation at station 1 (2) cannot have a causal effect on the data registered at station 2 (1) [53]. As the photon arrives at station $j = 1, 2$ it first passes through an electro-optic modulator (EOM) which rotates the polarization of the photon by an angle φ_j depending on the voltage applied to the EOM [53, 54]. This voltage is controlled by a binary variable A_j, which is chosen at random [53, 54]. Optionally, a bias voltage is added to the randomly varying voltage [53, 54]. The relation between the voltage applied to the EOM and the resulting rotation of the polarization is determined

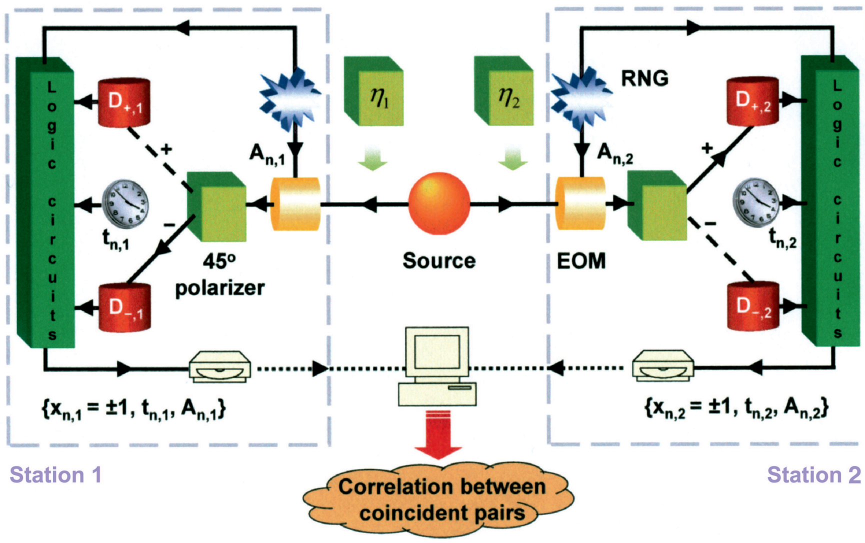


Fig. 2: (Color online) Schematic diagram of the EPRB experiment with photons [53, 54]. The source emits pairs of photons. One of the photons moves to station 1 and the other one to station 2. The photons in each pair have orthogonal but otherwise random polarizations. As the photon arrives at station $j = 1, 2$ it first passes through an electro-optic modulator (EOM) which rotates the polarization of the photon by an angle φ_j depending on the voltage applied to the EOM. This voltage is controlled by a binary variable A_j, which is chosen at random. As the photon leaves the EOM, a polarizing beam splitter directs it to one of the two detectors $D_{\pm,j}$. The detector produces a signal $x_{n,j} = \pm 1$ where the subscript n labels the nth detection event. Each station has its own clock which assigns a time-tag $t_{n,j}$ to each detection signal. The data set $\{x_{n,j}, t_{n,j}, A_{n,j} \,|\, n = 1, \ldots, N_j\}$ is stored on a hard disk for each station. Long after the experiment is finished both data sets can be analyzed and among other things, two-particle correlations can be computed.

experimentally, hence there is some uncertainty in relating the applied voltage to the rotation angle [53, 54]. As the photon leaves the EOM, a polarizing beam splitter directs it to one of the two detectors. The detector produces a signal $x_{n,j} = \pm 1$ where the subscript n labels the nth detection event. Each station has its own clock which assigns a time-tag $t_{n,j}$ to each signal generated by one of the two detectors [53, 54]. Effectively, this procedure discretizes time in intervals, the width of which is determined by the time-tag resolution τ. In the experiment, the time-tag generators are synchronized before each run [53, 54].

The firing of a detector is regarded as an event. At the nth event at station j, the dichotomic variable $A_{n,j}$, controlling the rotation angle $\varphi_{n,j}$, the dichotomic variable $x_{n,j}$ designating which detector fires, and the time tag $t_{n,j}$ of the detection event are written to a file on a hard disk, allowing the data to be analyzed long after the experiment has terminated [53, 54]. The set of data collected at station j may be written as

$$\Upsilon_j = \{x_{n,j}, t_{n,j}, \varphi_{n,j} : n = 1, \ldots, N_j\} , \tag{10}$$

where we allow for the possibility that the number of detected events N_j at stations $i = 1, 2$ need not (and in practice is not) to be the same and we have used the rotation angle $\varphi_{n,j}$ instead of the corresponding experimentally relevant dichotomic variable $A_{n,j}$ to facilitate the comparison with the quantum theoretical description.

Data analysis procedure

A laboratory EPRB experiment requires some criterion to decide which detection events are to be considered as stemming from a single or two-particle system. In EPRB experiments with photons, the decision to consider detection events as stemming from a two-particle system is taken on the basis of coincidence in time [53, 69]. Here we adopt the procedure employed by Weihs *et al.* [53, 54] Coincidences are identified by comparing the time differences $t_{n,1} - t_{m,2}$ with a window W, [53, 54, 69] where $n = 1, \ldots, N_1$ and $m = 1, \ldots, N_2$. By definition, for each pair of rotation angles a_1 and a_2, the number of coincidences between detectors $D_{x,1}$ ($x = \pm 1$) at station 1 and detectors $D_{y,2}$ ($y = \pm 1$) at station 2 is given by

$$\begin{aligned} C_{xy} &= C_{xy}(a_1, a_2) \\ &= \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} \delta_{x,x_{n,1}} \delta_{y,x_{m,2}} \delta_{a_1,\varphi_{n,1}} \delta_{a_2,\varphi_{m,2}} \Theta(W - |t_{n,1} - t_{m,2}|) , \end{aligned} \tag{11}$$

where $\Theta(t)$ denotes the unit step function. In (11) the sum over all events has to be carried out such that each event (= one detected photon) contributes only once. Clearly, this constraint introduces some ambiguity in the counting procedure as there is a priori, no clear-cut criterion to decide which events at stations $j = 1$ and $j = 2$ should be paired. One obvious criterion might be to choose the pairs such that C_{xy} is maximum, but such a criterion renders the data analysis procedure (not the data production) acausal. It is trivial though to analyze the data generated by the experiment of Weihs *et al.* such that conclusions do not suffer from this artifact [63]. In general, the values for the coincidences $C_{xy}(a_1, a_2)$ depend on the time-tag resolution τ and the window W used to identify the coincidences.

The single-particle averages and correlation between the coincidence counts are defined by

$$
\begin{aligned}
E_1(a_1,a_2) &= \frac{\sum_{x,y=\pm1} xC_{xy}}{\sum_{x,y=\pm1} C_{xy}} = \frac{C_{++}-C_{--}+C_{+-}-C_{-+}}{C_{++}+C_{--}+C_{+-}+C_{-+}} \\
E_2(a_1,a_2) &= \frac{\sum_{x,y=\pm1} yC_{xy}}{\sum_{x,y=\pm1} C_{xy}} = \frac{C_{++}-C_{--}-C_{+-}+C_{-+}}{C_{++}+C_{--}+C_{+-}+C_{-+}} \\
E(a_1,a_2) &= \frac{\sum_{x,y=\pm1} xyC_{xy}}{\sum_{x,y=\pm1} C_{xy}} = \frac{C_{++}+C_{--}-C_{+-}-C_{-+}}{C_{++}+C_{--}+C_{+-}+C_{-+}}, \qquad (12)
\end{aligned}
$$

where the denominator $N_c = N_c(a_1,a_2) = C_{++}+C_{--}+C_{+-}+C_{-+}$ in (12) is the sum of all coincidences and, in general, varies with the settings (a_1,a_2).

Local-realistic treatments of the EPRB experiment assume that the correlation, as measured in the experiment, is given by [70]

$$
C_{xy}^{(\infty)}(a_1,a_2) = \sum_{n=1}^{N} \delta_{x,x_{n,1}}\delta_{y,x_{n,2}}\delta_{a_1,\theta_{n,1}}\delta_{a_2,\theta_{m,2}}\,, \qquad (13)
$$

which is obtained from (11) (in which each photon contributes only once) by assuming that $N = N_1 = N_2$, pairs are defined by $n = m$ and by taking the limit $W \to \infty$. However, the working hypothesis that the value of W should not matter because the time window only serves to identify pairs may not apply to real experiments. The analysis of the data of the experiment of Weihs *et al.* shows that the average time between pairs of photons is of the order of $30\,\mu$s or more, much larger than the typical values (of the order of a few nanoseconds) of the time-window W used in the experiments [54]. In other words, in practice, the identification of photon pairs does not require the use of W's of the order of a few nanoseconds.

4.2. Theory versus real data

Quantum theory predicts the expectation values $\widehat{E}_1$, $\widehat{E}_2$ and $\widehat{E}_{12}$ but cannot say anything about individual events [71, 72]. The fundamental question is therefore how data that is produced by a laboratory or computer experiment and comes in the form of individual events relates to the statistical results of quantum theory. The tremendous success of quantum theory in describing a large variety of experiments is not an excuse for assuming that Nature "knows" about the probability distributions of quantum theory, generates events accordingly and does so by means of a magic "random" process. Instead, to avoid logical contradictions and/or the need of introducing mystical elements, it is safer to take an agnostic standpoint, that is make no assumption for which there is no empirical evidence and then demonstrate that *all* the averages computed from the (experimental or computer generated) data comply with the predictions of quantum theory.

4.3. Event-based simulation

A discrete-event computer simulation model of a laboratory EPRB experiment should be a one-to-one copy of the experimental setup and, most importantly, should not rely on quantum theory. Moreover, the simulation should involve only Einstein-local, cause-and-effect processes: acausal processes, such as the one used to maximize the coincidence counts, are prohibited.

A minimal, discrete-event simulation model of the EPRB experiment by Weihs *et al.* requires a specification of the information carried by the particles, of the algorithm that simulates the source, the polarizers, the detectors, and of the procedure to analyze the data. Since in the above description of the experiment the orientation of the polarization vectors $\boldsymbol{P}_j = (\cos\zeta_j, \sin\zeta_j, 0)$ and the orientations of the optical axis of the polarizers $\boldsymbol{a}_j = (\cos a_j, \sin a_j, 0)$ for $j = 1, 2$ is limited to the xy-plane we omit the z-component in the simulation.

Source and particles

Each time, the source emits two particles which carry a vector

$$\boldsymbol{u}_{n,j} \; = \; (\cos(\xi_n + (j-1)\pi/2), \sin(\xi_n + (j-1)\pi/2)) \, ,$$

representing the polarization of photons. This polarization is completely characterized by the angle ξ_n and the direction $j = 1, 2$ in which the particle moves. A uniform pseudo-random number generator is used to pick the angle $0 \le \xi_n < 2\pi$. Clearly, the source emits two particles with a mutually orthogonal, hence correlated but otherwise random polarization. Note that in the simulation and unlike in the laboratory experiment, there is an ambiguity as to which two particles have been generated as a pair.

Electro-optic modulator (EOM)

The EOM in station $j = 1, 2$ rotates the polarization of the incoming particle by an angle φ_j, that is its polarization angle becomes

$$\xi'_{n,j} \; \equiv \; \mathrm{EOM}_j(\xi_n + (j-1)\pi/2, \varphi_j) \; = \; \xi_n + (j-1)\pi/2 - \varphi_j$$

symbolically. Mimicking the experiment of Weihs *et al.* in which φ_1 can take the values a_1, a'_1 and φ_2 can take the values a_2, a'_2, we generate two binary uniform pseudo-random numbers $A_j = 0, 1$ and use them to choose the value of the angles φ_j, that is $\varphi_1 = a_1(1 - A_1) + a'_1 A_1$ and $\varphi_2 = a_2(1 - A_2) + a'_2 A_2$.

Beam-splitting polarizer

In laboratory EPRB experiments with photons, various polarizers are interchangeable, at least in principle. Therefore, the algorithms to simulate them should be identical. The simulation model for a polarizing beam splitter is defined by the rule

$$x_{n,j} \;=\; \begin{cases} +1 & \text{if } r_n \le \cos^2 \xi'_{n,j} \\ -1 & \text{if } r_n > \cos^2 \xi'_{n,j}\,, \end{cases} \tag{14}$$

where $0 < r_n < 1$ are uniform pseudo-random numbers. The polarizer sends a photon with polarization $\boldsymbol{u}_n = (\cos\varphi_j, \sin\varphi_j)$ or $\boldsymbol{u}_n = (-\sin\varphi_j, \cos\varphi_j)$ through its output channel labelled by $+1$ and -1, respectively. It is easy to see that for fixed $\xi'_{n,i} = \xi'_i$, this rule generates events such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_{n,j} \;=\; \cos^2 \varphi_{n,j}\,,$$

with probability one, showing that the distribution of events complies with Malus law.

This simplified mathematical model suffices to simulate the EPRB experiment but cannot be used to simulate other optics experiments [10, 8]. However, more complicated models used to simulate the polarizing beam splitter in these other experiments can be used to simulate the EPRB experiment [10].

Time-tag model

As is well-known, as light passes through an EOM (which is essentially a tuneable wave plate), it experiences a retardation depending on its initial polarization and the rotation by the EOM. We assume that for each particle this time delay is represented by [73, 74]

$$t_{n,i} \;=\; \lambda(\xi'_{n,i}) r'_n\,, \tag{15}$$

which is distributed uniformly ($0 < r'_n < 1$ is a uniform pseudo-random number) over the interval $[0, \lambda(\xi'_{n,i})]$. For $\lambda(\xi'_{n,i}) = T_0 \sin^4 2\xi'_{n,i}$ this time-tag model, in combination with the model of the polarizing beam splitter, rigorously reproduces the results of quantum theory of the EPRB experiments in the limit $W \to 0$ [73, 74]. We therefore adopt the expression $\lambda(\xi'_{n,i}) = T_0 \sin^4 2\xi'_{n,i}$ leaving only T_0 as an adjustable parameter.

Detector

The detectors are ideal particle counters, producing a click for each incoming particle. Hence, we assume that the detectors have 100% detection efficiency. Note that this model for the detector is the simplest one can imagine. More complicated models involving thresholds may be used as well [75, 76, 10].

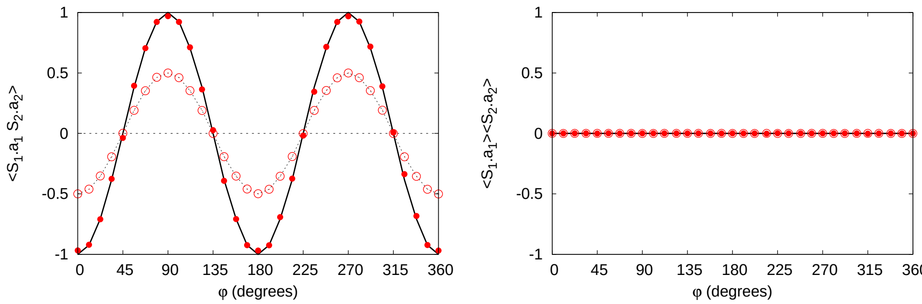


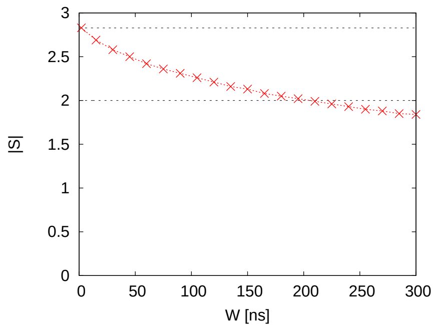
Fig. 3: (Color online) Simulation results (markers) and quantum theoretical result (solid line) for the EPRB experiment with $\varphi_1 = \varphi$ and $\varphi_2 = 0$ for the two-particle expectation value $E(\varphi) = \langle \boldsymbol{S}_1 \cdot \boldsymbol{a}_1 \boldsymbol{S}_2 \cdot \boldsymbol{a}_2 \rangle$ (left) and the product of the two single-particle expectation values $E_1(\varphi)E_2(\varphi) = \langle \boldsymbol{S}_1 \cdot \boldsymbol{a}_1 \rangle \langle \boldsymbol{S}_2 \cdot \boldsymbol{a}_2 \rangle$ (right) as a function of φ. The source emits two photons with orthogonal but otherwise random polarization. The number of emitted photon pairs $N = (N_1 + N_2)/2 = 10^6$ with $N_1 = N_2$ and the adjustable parameter in the time-tag model $T_0 = 10^3$. Solid circles: coincidence counting with $W/\tau = 1$; open circles: no coincidence counting.

Simulation and data analysis procedure

The simulation algorithm generates the data sets Υ_i, similar to the ones obtained in the experiment (see (10)). In the simulation we have $N_1 = N_2$. We analyze the data sets in exactly the same manner as the experimental data are analyzed, implying that we include the post-selection procedure to select photon pairs by a time-coincidence window W. In order to count the coincidences, we choose a time-tag resolution $0 < \tau < T_0$ and a coincidence window $\tau \leq W$. We set the correlation counts $C_{xy}(\varphi_1, \varphi_2)$ to zero for all $x, y = \pm 1$. We compute the discretized time tags $k_{n,j} = \lceil t_{n,j}/\tau \rceil$ for all events in both data sets. Here $\lceil x \rceil$ denotes the smallest integer that is larger or equal to x, that is $\lceil x \rceil - 1 < x \leq \lceil x \rceil$. According to the procedure adopted in the experiment [53, 54], an entangled photon pair is observed if and only if $|k_{n,1} - k_{n,2}| < k = \lceil W/\tau \rceil$. Thus, if $|k_{n,1} - k_{n,2}| < k$, we increment the count $C_{x_{n,1}, x_{n,2}}(\varphi_1, \varphi_2)$. Although in the simulation the ratio of detected to emitted photons is equal to one, the final detection efficiency is reduced due to the time-coincidence post-selection procedure.

4.4. Simulation results

Figure 3 presents the simulation results (markers) with $\varphi_1 = \varphi$ and $\varphi_2 = 0$ for the two-particle expectation value $E(\varphi) = \langle \boldsymbol{S}_1 \cdot \boldsymbol{a}_1 \boldsymbol{S}_2 \cdot \boldsymbol{a}_2 \rangle$ (left) and the



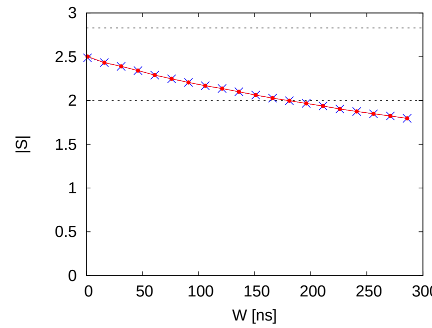


Fig. 4: (Color online) The Bell-CHSH function $|S| = |E(a,b) - E(a,b') + E(a',b) + E(a',b')|$ as a function of the time window W for $a=0$, $a'=\pi/4$, $b=\pi/8$ and $b'=3\pi/8$. The dashed lines represent the maximum value for a quantum system of two $S=1/2$ particles in a separable (product) state ($|S|=2$) and in a singlet state ($|S|=2\sqrt{2}$), respectively. If $|S|>2$, the Bell-CHSH inequality is violated. Left: Red crosses: discrete-event simulation data. The total number of pairs generated is 3×10^5 (roughly the same as in experiment [53]) and $T_0 = 2000\,\text{ns}$ (see (15)). Right: Results extracted from the data set "newlongtime2" recorded in experiments performed by Weihs [53, 54]. Red bullets connected by the red solid line: results obtained by maximizing the coincidence counts through a time-shift $\Delta = 0.5\,\text{ns}$. Blue crosses: $\Delta = 0$. For $W \approx 200\,\text{ns}$, well below the average time between pair (about $30\,\mu\text{s}$) the data does not violate the Bell-CHSH inequality ($|S| \le 2$). For $W > 20\,\text{ns}$ the change of some of these single-spin averages (data not shown) observed by Bob (Alice) when Alice (Bob) changes her (his) setting, systematically exceeds five standard deviations, suggesting it is highly unlikely that the data is in concert with quantum theory of the EPRB experiment [64].

product of the two single-particle expectation values

$$E_1(\varphi)E_2(\varphi) \;=\; \langle \boldsymbol{S}_1 \cdot \boldsymbol{a}_1 \rangle \langle \boldsymbol{S}_2 \cdot \boldsymbol{a}_2 \rangle$$

(right) as a function of φ. Both the results of a coincidence counting analysis (solid markers) and of the analysis without coincidence counting (open markers) is shown. The results expected from the quantum theoretical description are $\widehat{E}(\varphi,0) = -\cos 2\varphi$ and $\widehat{E}_1(\varphi) = \widehat{E}_2(0) = 0$ and are represented by the solid lines. The coincidence counting analysis with $W/\tau = 1$ (solid markers), which is similar to the one used in the experiment by Weihs *et al.* [53, 54], gives results which fit very well to the prediction of quantum theory.

When all emitted particles are taken into account (open markers), corresponding to a data analysis procedure without using a time-coincidence window W to select pairs, $E(\varphi,0) = -(\cos 2\varphi)/2 = \widehat{E}(\varphi,0)/2$. Note that this data analysis procedure is equivalent to a procedure in which $W \to \infty$ or to a procedure in which the time-tag data is simply omitted. The difference with the case of coincidence counting with a finite W demonstrates the fact that the observation of two-particle correlations $\rho_{12}(\varphi, 0)$ corresponding to those of the singlet state $\widehat{\rho}_{12}(\varphi, 0) = -\cos 2\varphi$ strongly depends on how the data is analyzed (size of the time-coincidence window) and not only on how the photon pairs are generated, which in the case of our computer experiments is "on demand".

In summary, these simulation results leave no room for an interpretation other than the "non-quantum", discrete-event simulation of a real laboratory experiment reproduces *all* the results of quantum theory of two photons in a polarization-entangled singlet state. As the simulation results are, for all practical purposes, the same as those of quantum theory, it is no surprise that the "non-quantum" simulation model also violates Bell inequalities. Furthermore, as the algorithm simulates Einstein-local processes only and does not contain operations that mimic "action-on-a-distance" the simulation also proves (via the computer-mechanical device metaphore) that there is no need to invoke the latter in order to give a rational "classical-statistical" explanation for the correlations of a quantum system in the singlet state.

As shown in Fig. 4, the statement that the simulation data violates the Bell-CHSH inequality $|S| > 2$ strongly depends on the time window W. While in the case of the laboratory experiment, one might argue that as W increases the identification of pairs becomes less certain, resulting in a decrease of $|S|$, taking into account the time scales involved ($W \approx 300\,\text{ns}$ versus an average time of $30\,\mu\text{s}$ between single events), this argument is not very convincing. Indeed, the "perfect" experiment carried out on a digital computer shows the same dependence of $|S|$ on W without suffering from the pair-identification problem, see Fig. 4.

4.5. Violating Bell inequalities by locally causal processes

It is of interest to examine the mechanism by which the discrete-event model (and the laboratory experiments) produce data that leads to a violation of inequalities such as $|S| \leq 2$, even though every pair of particles that arrives at the detectors is accounted for. In essence, the reason is simple: it is the coincidence window that acts as a filter, removing pairs for which the difference of the time-delays incurred while passing through the optical components is larger than W. As these time-delays depend on the settings of these components (once more a metaphor for the interactions of the photons with the optical components), the coincidence counts depend on these settings. The idea that one can carry out a laboratory experiment and identitify pairs without a proper space-time labelling will remain a pipe-dream as long as the man-made world of mathematical models and concepts differs from the world which we experience. In other words, when the aim is to construct simulation models that, through the computer-experiment metaphor, represent real laboratory experiments, it is necessary to include *all* the elements that are essential to the laboratory experiments. Otherwise, the relation between computer and laboratory experiment is lost.

We use an abstraction of the computer simulation algorithm in terms of a Kolmogorovian probabilistic model to scrutinize in depth the mechanism by which the time-coincidence window leads to violations of Bell-like inequalities. In [74, 6], we present such a probabilistic description and (i) rigorously proved that for up to first order in W the model yields the single particle averages and the two-particle correlations of quantum theory for the system under consideration; (ii) discuss how the presence of the time-window W introduces correlations that cannot be described by the original Bell-like "hidden-variable" models [70] and how the non-Kolmogorovian nature appears. Here, we closely follow [6].

The time-coincidence post-selection procedure with the time-window W defines the "coincident" photons based on the time-tags $t_{n,i}$ thereby reducing the final detection efficiency to less than 100%, although in the simulation a measurement always returns a $+1$ or -1 for both photons in a pair (100% detection efficiency of the detectors). Hence, even in case of a perfect detection process the data set that is finally retained consists only of a subset of the entire ensemble of correlated photons that was emitted by the source, exactly as in the laboratory experiments [53].

In the spirit of Kolmogorov probability theory, let us assume that there exists a probability $P(x_1, x_2, t_1, t_2|\varphi_1, \varphi_2)$ to observe the data $\{x_i, t_i\}$ conditional on the settings φ_i at stations i for $i = 1, 2$. The probability can always be expressed as an integral over the mutually exclusive events ξ_1, ξ_2, representing the polarization of the photons

$$\begin{aligned} P(x_1, x_2, t_1, t_2|\varphi_1, \varphi_2) \quad &= \quad \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} P(x_1, x_2, t_1, t_2|\varphi_1, \varphi_2, \xi_1, \xi_2) \\ &\quad \times P(\xi_1, \xi_2|\varphi_1, \varphi_2) d\xi_1 d\xi_2 \,. \qquad (16) \end{aligned}$$

We now assume that in the probabilistic version of our simulation model, for each event, (i) the values of $\{x_1, x_2, t_1, t_2\}$ are independent of each other, (ii) the values of $\{x_1, t_1\}$ ($\{x_2, t_2\}$) are independent of φ_2 and ξ_2 (φ_1 and ξ_1), (iii) ξ_1 and ξ_2 are independent of φ_1 or φ_2. With these assumptions Eq. (16) becomes

$$\begin{aligned} P(x_1, x_2, t_1, t_2|\varphi_1, \varphi_2) \quad &= \quad \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} P(x_1|\varphi_1, \xi_1) P(t_1|\varphi_1, \xi_1) P(x_2|\varphi_2, \xi_2) \\ &\quad \times P(t_2|\varphi_2, \xi_2) P(\xi_1, \xi_2) d\xi_1 d\xi_2 \,. \qquad (17) \end{aligned}$$

It is not difficult to see that (17) is the generic probabilistic description of the simulation algorithm presented in Sect. 4.3 if we leave out the detailed specification of the operation of the optical components.

In the experiment [53] and therefore also in our simulation model, the events are selected using a time window W. Accounting for the time window, that is multiplying (17) by a step function and integrating over all t_1 and t_2, the expression for the probability for observing the event (x_1, x_2) reads

$$P(x_1, x_2|\varphi_1, \varphi_2) = \int_0^{2\pi}\int_0^{2\pi} P(x_1|\varphi_1, \xi_1)P(x_2|\varphi_2, \xi_2)\rho(\xi_1, \xi_2|\varphi_1, \varphi_2)d\xi_1 d\xi_2 \,, \tag{18}$$

where the probability density $\rho(\xi_1, \xi_2|\varphi_1, \varphi_2)$ is given by

$$\rho(\xi_1, \xi_2|\varphi_1, \varphi_2) = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(t_1|\varphi_1, \xi_1)P(t_2|\varphi_2, \xi_2)\varphi(W - |t_1 - t_2|)P(\xi_1, \xi_2)dt_1 dt_2}{\int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(t_1|\varphi_1, \xi_1)P(t_2|\varphi_2, \xi_2)\varphi(W - |t_1 - t_2|)P(\xi_1, \xi_2)d\xi_1 d\xi_2 dt_1 dt_2} \,. \tag{19}$$

From (19) it follows immediately that in general, $\rho(\xi_1, \xi_2|\varphi_1, \varphi_2)$ depends on the settings φ_1 and φ_2. This fact brings the derivation of the original Bell (CHSH) inequality to a halt. Indeed, in these derivations it is assumed that the probability distribution for ξ_1 and ξ_2 does not depend on the settings φ_1 or φ_2 [72, 70]. From (18) and (19) it is also clear that, in general, the presence of the time window renders the probabilistic description of the event (x_1, x_2) non-Kolmogorovian: the probability density (19) depends on the settings φ_1 or φ_2. By ignoring the time-tag information ($W \to \infty$), $\rho(\xi_1, \xi_2|\varphi_1, \varphi_2)$ becomes independent of φ_1 and φ_2 and the two-particle probability takes the form of the hidden variable models considered by Bell [70].

Summarizing: Our simulation model and its probabilistic version (17) describe local processes only. It is the filtering of the detection events by means of the time-coincidence window W produce correlations which violate Bell-type inequalities [77, 78]. For $W \to 0$ the classical, local and causal simulation model and its probabilistic abstraction yields single-particle and two-particle averages that are the same as those of a singlet state in quantum theory.

5. Discussion

We have discussed a computer simulation approach to model quantum physics phenomena without making use of quantum theory. Specifically, in this paper, we have shown that the event-based simulation model provides a cause-and-effect description of laboratory EPRB experiments at a level of detail which is not covered by quantum theory, such as the effect of the choice of the time-window W on the coincidence counts. The statistical distributions which ensue and which are usually thought to be of quantum mechanical origin, emerge from a time series of discrete events generated by Einstein-local, cause-and-effect processes which in principle could be executed using a macroscopic mechanical computer. The violations of the Bell-type inequalities is a direct consequence of the use of a coincidence criterion which is unavoidable to identify pairs of particles in space-time. The post-selection procedure is essential for any EPRB experiment involving pairs of objects but it is not required for the discrete-event simulation of Bell-inequality tests with neutrons [79] in which the path and the magnetic moment of the neutron are correlated, see [7].

Invoking the computer-mechanical device metaphor mentioned in the Introduction, it is clear that whatever algorithm the digital computer carries out, there exists a one-to-one mapping to objects that are directly accessible to our senses. Therefore computer simulation offers unique possibilities to confront man-made concepts and theories with actual facts, not just abstract symbols, facilitating the ordering and deeper understanding of human experience.

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