

Nonperturbative infrared dynamics of three-dimensional QED with a four-fermion interaction

Valery Gusynin

Bogolyubov Institute for Theoretical Physics, 03143 Kiev, Ukraine

Anthony Hams and Manuel Reenders

Institute for Theoretical Physics, University of Groningen, 9747 AG Groningen, The Netherlands

(Received 31 May 2000; published 30 January 2001)

A nonlinear Schwinger-Dyson (SD) equation for the gauge boson propagator of massless QED in one time and two spatial dimensions is studied. It is shown that the nonperturbative solution leads to a nontrivial renormalization-group infrared fixed point quantitatively close to the one found in the leading order of the $1/N$ expansion, with N the number of fermion flavors. In the gauged Nambu–Jona-Lasinio (GNJL) model an equation for the Yukawa vertex is solved in an approximation given by the one-photon exchange and an analytic expression is derived for the propagator of the scalar fermion-antifermion composites. Subsequently, the mass and width of the scalar composites near the phase transition line are calculated as functions of the four-fermion coupling g and flavor number N . The possible relevance of these results for describing particle-hole excitations, in particular antiferromagnetic correlations, observed in the underdoped cuprates, is briefly discussed.

DOI: 10.1103/PhysRevD.63.045025

PACS number(s): 11.10.Hi, 11.10.St, 11.30.Qc, 11.30.Rd

I. INTRODUCTION

Quantum electrodynamics in $2+1$ dimensions (QED₃) has attracted much interest over recent years. Its version with N flavors of massless four-component Dirac fermions shares a number of features, such as confinement and chiral symmetry breaking, with four-dimensional quantum chromodynamics (QCD). The loop expansion of a massless theory suffers from severe infrared divergencies. However, in the $1/N$ expansion, the theory becomes infrared finite [1], with the effective dimensionless coupling

$$\bar{\alpha}(p) = \frac{e^2}{p[1+\Pi(p)]}, \quad \Pi(p) = \frac{e^2 N}{8p}, \quad p = \sqrt{p^2}, \quad (1)$$

giving rise to the renormalization-group β function

$$\beta_{\bar{\alpha}}(\bar{\alpha}) \equiv p \frac{d\bar{\alpha}(p)}{dp} = -\bar{\alpha} \left(1 - \frac{N}{8\bar{\alpha}} \right). \quad (2)$$

In Eq. (1) e is the dimensionful gauge coupling and $\Pi(p)$ is the polarization operator. At large momenta ($p \gg \alpha \equiv e^2 N/8$) the effective coupling (1) approaches zero (asymptotic freedom) while for small momenta ($p \ll \alpha$) it runs to the infrared (IR) fixed point $8/N$. Here, the dimensionful parameter α plays a role similar to the Λ_{QCD} scale. Since QED₃ is a super-renormalizable theory, the running of the coupling should be understood as a Wilsonian rather than Gell-Mann–Low type, and it is not associated with ultraviolet divergencies.

By studying the Schwinger-Dyson (SD) equation for the fermion self-energy in leading order of the $1/N$ expansion, it was found in Ref. [2] that a phase transition occurs when the coupling at the IR fixed point exceeds some critical value ($8/N > \pi^2/4$). This means there exists a critical number of fermions N_{cr} ($N_{cr} = 32/\pi^2 \approx 3.24$) below which dynamical mass generation takes place and above which the fermions remain massless. This is similar to what happens in quenched QED₄ [3,4], where the gauge coupling must ex-

ceed a critical value for chiral symmetry breaking to occur. The appearance of a dimensionless critical coupling can be traced to the scale invariant behavior of both theories. The scale invariance of QED₃ is associated with the IR fixed point, since, as is evident from Eq. (1), in the limit $p \ll \alpha$ the dimensional parameter e drops out of the running coupling (as well as from SD equations for the Green's functions). Related to this is the fact that the chiral symmetry breaking phase transition in both theories belongs to a special universality class called conformal phase transition (CPT) introduced in Ref. [5]. It is characterized by a scaling function having an essential singularity at the transition point, and by abrupt change of the spectrum of light excitations as the critical point is crossed (for details about the CPT in QED₃ see Ref. [6]).

The presence of a critical N_{cr} in QED₃ is intriguing especially because of possible existence of an analogous critical fermion number $N_f = N_{cr}$ in $(3+1)$ -dimensional $SU(N_c)$ gauge theories, as is suggested by both analytical studies [7,8,5] and lattice computer simulations [9,10]. Also, a nontrivial IR fixed point in QED₃ may be related to nonperturbative dynamics in condensed matter, in particular, dynamics of non-Fermi liquid behavior [11,12].

The fact that the value of the IR fixed point determines the critical N_{cr} , below which the system is in the symmetry broken phase, and that this critical value is found to be of order 3 provides motivation for searches beyond the $1/N$ expansion. It is especially important because there is still controversy concerning the existence of finite N_{cr} in QED₃; some authors argue that the generation of a fermion mass occurs at all values of N [13,14] what might mean the absence of the IR fixed point for the running coupling.¹ Despite the arguments of Ref. [15] in addition to the fact that studies

¹This would happen, for example, if one finds more soft behavior of the polarization operator in the infrared, like $\Pi(p) \sim (\alpha/p)^\gamma$, with $\gamma < 1$.

of $1/N^2$ corrections to the gap equation showed the increase of the critical value ($N_{\text{cr}} = 128/3\pi^2 \simeq 4.32$) [16–18], the situation is far from being conclusive. What we need is some kind of self-consistent equation for the running coupling which is to be solved nonperturbatively.

In the present paper we study such a nonlinear equation for the running coupling which is the analogue of the ladder approximation for the fermion propagator.² Similar to the gap equation, the kernel is taken in the $1/N$ approximation, where it is nothing else as the one-loop photon-photon scattering amplitude with zero momentum transfer. The equation obtained is obviously gauge invariant. We then study our equation both analytically and numerically. We find that the vacuum polarization operator, obtained as a nonperturbative solution of the equation, has the same infrared asymptotics as the one-loop expression: $\Pi(p) \simeq C\alpha/p$, $C \simeq 1 + 1/14N$. Thus a nontrivial IR fixed point persists in the nonperturbative solution. Moreover, the correction to the one-loop result ($C=1$) is small even at $N=1$ due to smallness of the numerical coefficient before $1/N$, that explains why the leading order in the $1/N$ expansion (the one-loop approximation) for the vacuum polarization works so well.

Further we proceed to studying QED₃ with additional four-fermion interactions [the gauged Nambu–Jona-Lasinio (GNJL) model]. Such kind of models are considered to be effective theories at long distances in planar condensed matter physics, in particular, for high temperature superconductivity [20]. It is well known that in the improved ladder approximation (with the photon propagator including fermion one-loop effects) this model has a nontrivial phase structure [21] in the coupling constant plane ($1/N, g$), where $g = 2G\Lambda/\pi^2$ is the dimensionless four-fermion coupling (Λ is the ultraviolet cutoff). The critical line is

$$g_c(1/N) = \frac{1}{4} \left(1 + \sqrt{1 - \frac{N_{\text{cr}}}{N}} \right)^2, \quad N > N_{\text{cr}}, \quad (3)$$

at $g > 1/4$, and $1/N = 1/N_{\text{cr}}$ at $g < 1/4$. Above this line the gap equation for the fermion self-energy $\Sigma(p)$ has a nontrivial solution. Thus the chiral symmetry is dynamically broken, which implies the existence of a nonzero vacuum condensate $\langle \bar{\psi}\psi \rangle$. One end point ($1/N=0, g=1$) of the critical line corresponds to the ordinary NJL model (in $2+1$ dimensions), while the other one ($1/N=1/N_{\text{cr}}, g=0$) corresponds to pure QED₃.

A nice feature of this model is that it is renormalizable in the $1/N$ expansion [22] leading to an interacting continuum ($\Lambda \rightarrow \infty$) theory near a critical scaling region (critical curve) separating a chiral symmetric phase (χS) and a spontaneous chiral symmetry broken phase ($S\chi SB$). The spectrum of such a theory contains pseudoscalar (π) and scalar (σ)

bound states which become light in the vicinity of the critical line. Since the phase transition is second order along the $g > 1/4$ part Eq. (3) of the critical curve, scalar and pseudo-scalar resonances are to be produced on the symmetric side of the curve, whose masses approach zero as the critical curve is approached [23,24]. The part of the critical curve with $g < 1/4, 1/N = 1/N_{\text{cr}}$ is rather special and is related to the CPT in pure QED₃ (we shall discuss it more in the main text).

In this work we study scalar composites (σ and π bosons) which are resonances in the symmetric phase of the $2+1$ -dimensional GNJL model. The π boson can be viewed as a Goldstone boson precursor mode that comes down in energy as the transition is approached. Our study is motivated partially by possible relation of these resonances to spin excitations observed in neutron scattering experiments in underdoped high- T_c superconductors [25]. We calculate their masses and widths as a function of the four-fermion coupling g and therefore mass and width's dependence on the doping concentration (since in certain low-energy effective models based on spin-charge separation, the coupling g would depend on the doping, e.g., Ref. [26]).

The plan of the present paper is as follows. In Sec. II we derive a nonlinear equation for the effective running coupling in pure QED₃ which is then solved both analytically and numerically to establish the existence of a nontrivial IR fixed point. In Sec. III after introducing the GNJL model in $2+1$ dimensions we solve the equation for the Yukawa vertex with nonzero boson momentum. In Sec. IV we obtain an analytical expression for the boson propagator valid along the entire critical line and analyze its behavior in different asymptotical regimes. The analysis of the scalar composites near the critical line (3) is given in Sec. V. We present our summary in Sec. VI. In Appendix A we compute the one-loop photon-photon scattering amplitude with zero transferred momentum and list some useful angular integrals. In Appendix B, an expression for the nonlocal gauge for the ladder (bare vertex) approximation is derived. Finally, in Appendix C, we present some details of the approximation [24] which is used to solve the equation for the Yukawa vertex.

II. THE EQUATION FOR THE RUNNING COUPLING IN QED₃

The Lagrangian density of massless QED₃ in a general covariant gauge is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2a}(\partial_\mu A^\mu)^2 + \bar{\psi}_i i \gamma^\mu D_\mu \psi_i, \quad (4)$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative. In a parity invariant formulation we consider N flavors of fermions ($i = 1, \dots, N$) described by four-component spinors. The three 4×4 γ matrices are taken to be

²Recently, in Ref. [19] another nonlinear equation for the running coupling was proposed in order to study nontrivial infrared structure of the theory. However, their definition of the running coupling deviates considerably from the standard one used in the present paper and we will not attempt to compare both approaches.

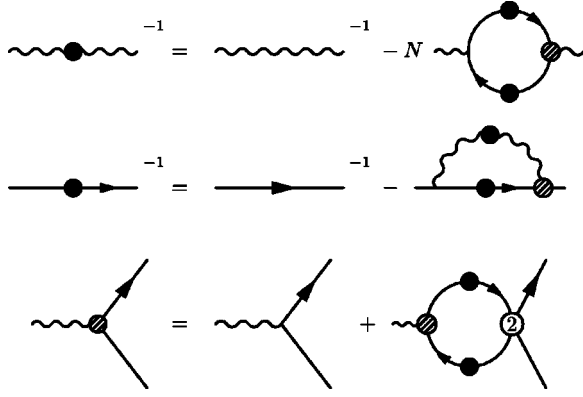


FIG. 1. Exact SD equations for the gauge boson propagator $D_{\mu\nu}$, the fermion propagator S , and the vertex Γ^μ .

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad (5)$$

with σ_i the usual Pauli matrices. There are two matrices,

$$\gamma^3 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma_5 = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (6)$$

that anticommute with γ^0 , γ^1 , and γ^2 . Therefore for each four-component spinor, there is a global $U(2)$ symmetry with generators I , $\frac{1}{i}\gamma^3$, γ^5 , and $\frac{1}{2}[\gamma^3, \gamma^5]$, and the full symmetry is then $U(2N)$. In what follows we shall restrict ourselves to the symmetric phase of the model, i.e., massless fermions.

The exact SD equations are given in Fig. 1. For clarity we have extracted the explicit factors of N coming from the one-fermion loop. Since in pure QED_3 we have only one dimensionful parameter e , this enables us to choose our scale such that Ne^2 remains fixed. This means that every photon propagator (times e^2) contributes one factor of $1/N$.

To make a $1/N$ expansion of Fig. 1, we first need to expand the two-fermion, one-photon irreducible fermion-fermion scattering kernel, see Fig. 2. We can convince ourselves that Fig. 2 is indeed the right expansion, since the only corrections of order one are fermion loops and they are already included in the full photon propagator. Inserting this expansion into the SD equation for the vertex, we obtain a closed set of integral equations. The nice feature of this truncated system of SD equations is that it satisfies the Ward-Takahashi (WT) identities for the vertex as well as for the vacuum polarization [18]. However, finding an analytic solution seems to be a formidable task and further approxima-

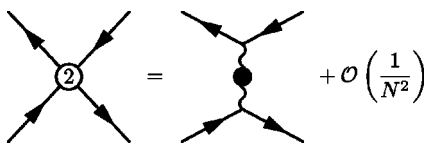


FIG. 2. The $1/N$ expansion of the two-fermion, one-photon irreducible fermion-fermion scattering kernel.

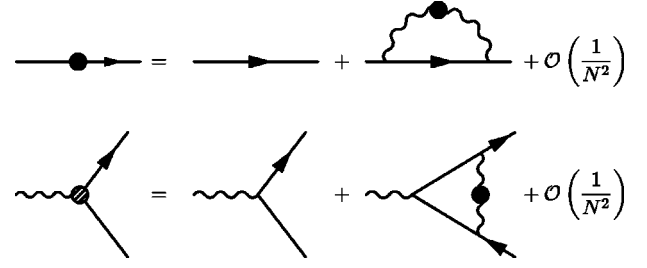


FIG. 3. SD equation for fermion propagator and vertex up to order $1/N^2$.

tions are required. For such an approximation, we simplify the fermion and the vertex SD equations by keeping only the lowest order terms in the $1/N$ expansion (see Fig. 3). Then, inserting this into the SD equation for the photon propagator we obtain the equation shown in Fig. 4 which still satisfies the WT identity. After that equation has been solved, the fermion propagator and the vertex can be evaluated explicitly through right-hand sides of Fig. 3.

We could solve the obtained photon propagator equation by further iteration, with the one-loop fermion correction included at the initial step to obtain a perturbative $1/N$ expansion. Instead we choose to solve the nonlinear integral equation given by Fig. 4 as it is. In this way we might get a hint of any nonanalytic behavior in $1/N$ which would be lost otherwise. At first glance, this way of solving a truncated system of SD equations ignores possible nonanalyticity in $1/N$ coming from the fermion wave function and the vertex (related, for example, to the power-law behavior due to an anomalous dimension). Note, however, that the fermion propagator is a gauge dependent quantity, thus possible power-law behavior of the fermion wave function must cancel the corresponding behavior coming from the longitudinal part of the vertex [recall that we consider in this paper the symmetric (massless) phase only]. Therefore the only nonanalyticity we have neglected is the one which might be present in the transverse part of the vertex beyond order $1/N$. Neglecting possible nonanalyticity in the transverse vertex means that we are seeking for nonanalyticity originating from the nonlinear equation for the photon propagator only. To some extent, the considered approximation is similar to solving the SD equation for the fermion mass function in the ladder approximation, where the photon propagator is taken in the leading $1/N$ order [1].

In $2+1$ dimensions, the SD equation for the photon propagator reads

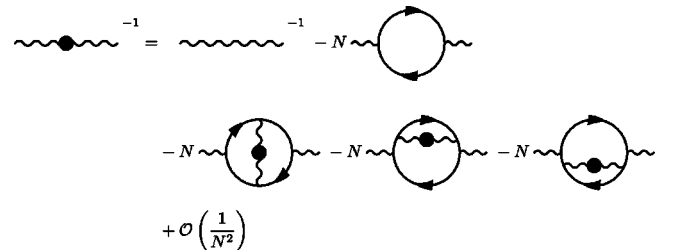
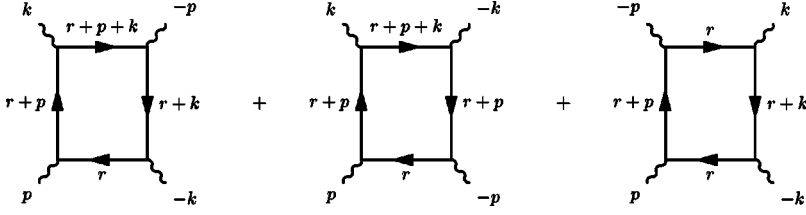


FIG. 4. Closed SD equation for the gauge boson propagator in next-to-leading $1/N$ expansion.

FIG. 5. The box diagram $B^{\mu\rho\nu\sigma}$.

$$D_{\mu\nu}^{-1}(p) = D_{0\mu\nu}^{-1}(p) + \Pi_{\mu\nu}(p), \quad (7)$$

with $D_{0\mu\nu}$ the bare photon propagator, and where $\Pi_{\mu\nu}$ is the vacuum polarization tensor

$$\Pi^{\mu\nu}(p) = iNe^2 \int_{M(2\pi)^3} \frac{d^3r}{(2\pi)^3} \text{Tr}[\gamma^\mu S(r+p) \Gamma^\nu(r+p, r) S(r)]. \quad (8)$$

Because of the gauge symmetry the vacuum polarization tensor is transverse:

$$\Pi^{\mu\nu}(p) = (-g^{\mu\nu}p^2 + p^\mu p^\nu) \Pi(p), \quad (9)$$

therefore, for the full photon propagator in a general covariant gauge, we have

$$D_{\mu\nu}(p) = \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2 [1 + \Pi(p)]} - a \frac{p_\mu p_\nu}{p^4}. \quad (10)$$

Moreover, one can write

$$\Pi(p) = -\frac{1}{2p^2} \left(g_{\mu\nu} - c_1 \frac{p_\mu p_\nu}{p^2} \right) \Pi^{\mu\nu}(p), \quad (11)$$

where the constant c_1 can be chosen arbitrarily.

The vacuum polarization $\Pi(p)$ governs the running of the dimensionless gauge coupling. Now we study the integral equation based on Fig. 4, this gives

$$\Pi^{\mu\nu}(p) = \Pi_1^{\mu\nu}(p) + \Pi_2^{\mu\nu}(p) + \mathcal{O}(1/N), \quad (12)$$

where $\Pi_1^{\mu\nu}(p)$ is the one-loop vacuum polarization,

$$\Pi_1^{\mu\nu}(p) = iNe^2 \int_{M(2\pi)^3} \frac{d^3r}{(2\pi)^3} \text{Tr}[\gamma^\mu S_0(r+p) \gamma^\nu S_0(r)], \quad (13)$$

with $S_0(p)$ the bare fermion propagator, $S_0(p) = 1/\hat{p}$, and

$$\Pi_2^{\mu\nu}(p) = iNe^4 \int_{M(2\pi)^3} \frac{d^3k}{(2\pi)^3} D_{\rho\sigma}(k) B^{\mu\rho\nu\sigma}(p, k), \quad (14)$$

where $B^{\mu\rho\nu\sigma}(p, k)$ is the one-loop ‘‘photon-photon’’ scattering amplitude, with zero momentum transfer, i.e.,

$$\begin{aligned} B^{\mu\rho\nu\sigma}(p, k) &= i \int_{M(2\pi)^3} \frac{d^3r}{(2\pi)^3} \text{Tr}[\gamma^\mu S_0(r+p) \gamma^\rho S_0(r+p+k) \\ &\quad \times \gamma^\nu S_0(r+k) \gamma^\sigma S_0(r)] \\ &\quad + i \int_{M(2\pi)^3} \frac{d^3r}{(2\pi)^3} \text{Tr}[\gamma^\mu S_0(r+p) \gamma^\rho S_0(r+p+k) \\ &\quad \times \gamma^\sigma S_0(r+p) \gamma^\nu S_0(r)] \\ &\quad + i \int_{M(2\pi)^3} \frac{d^3r}{(2\pi)^3} \text{Tr}[\gamma^\mu S_0(r+p) \gamma^\nu S_0(r) \gamma^\rho \\ &\quad \times S_0(r+k) \gamma^\sigma S_0(r)]. \end{aligned} \quad (15)$$

A graphical representation of the ‘‘box’’ diagram (15) in terms of Feynman diagrams is given in Fig. 5.

For the scattering amplitude $B^{\mu\rho\nu\sigma}$ there exists a Ward-Takahashi identity [27], which states the transversality of the amplitude with respect to external photon momenta,

$$p^\mu B_{\mu\nu\rho\sigma}(p, k, q, r) = 0, \quad k^\nu B_{\mu\nu\rho\sigma}(p, k, q, r) = 0, \quad \text{etc.} \quad (16)$$

The vacuum polarization tensor has a superficial linearly divergent part, which can be removed by a proper gauge-invariant regularization. However, since the divergent part is proportional to $g_{\mu\nu}$ we can project out the finite vacuum polarization by contracting $\Pi_{\mu\nu}(p)$ with the projector

$$P_{\mu\nu}(p) = \left(g_{\mu\nu} - 3 \frac{p_\mu p_\nu}{p^2} \right), \quad (17)$$

i.e., we choose the constant c_1 in Eq. (11) to be $c_1 = 3$. This approach was used in Refs. [28] and [15]. In this way, we obtain

$$\Pi(p) = \Pi_1(p) + \Pi_2(p) + \mathcal{O}(1/N), \quad (18)$$

with

$$\Pi_1(p) = -\frac{4iNe^2}{p^2} \int_{M(2\pi)^3} \frac{d^3k}{(2\pi)^3} \left[\frac{k^2 - 2k \cdot p - 3(k \cdot p)^2/p^2}{k^2(k+p)^2} \right], \quad (19)$$

$$\Pi_2(p) = \frac{iNe^4}{2p^2} \int_{M(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{B(p^2, k^2, p \cdot k)}{k^2 [1 + \Pi(k)]}, \quad (20)$$

where

$$B(p^2, k^2, p \cdot k) = g_{\mu\nu} g_{\rho\sigma} B^{\mu\rho\nu\sigma}(p, k). \quad (21)$$

In Euclidean formulation the above expressions can be written as

$$\begin{aligned}\Pi_1(p) &= \frac{2Ne^2}{\pi^2 p} \int_0^\infty dk \int \frac{d\Omega}{4\pi} \left[\frac{k^2 - 2k \cdot p - 3(k \cdot p)^2/p^2}{p(k+p)^2} \right] \\ &= \frac{Ne^2}{8p},\end{aligned}\quad (22)$$

$$\Pi_2(p) = -\frac{Ne^4}{4\pi^2 p} \int_0^\infty dk \frac{K(p,k)}{p[1+\Pi(k)]}, \quad (23)$$

where

$$K(p,k) \equiv \int \frac{d\Omega}{4\pi} B(-p^2, -k^2, -p \cdot k). \quad (24)$$

From Figs. 4 and 5, one can see that the first term in Eq. (15) corresponds to a vertex correction and the last two terms are fermion self-energy corrections. The sum of these diagrams has symmetries which provide a consistency check on the final result. From the graphical representation it is obvious that the quantity $B(p^2, k^2, p \cdot k)$ should be invariant under $p \leftrightarrow k$ and under $p \rightarrow -p$ or $p \cdot k \rightarrow -p \cdot k$.

A detailed computation of the ‘‘box’’ function B is presented in Appendix A, and the final expression for B is given by Eq. (A11). One can verify that Eq. (A11) has the symmetries we mentioned above. Finally, we perform the angular integration to obtain $K(p,k)$,

$$\begin{aligned}K(p,k) &= \mathcal{P} \int \frac{d\Omega}{4\pi} \left[\frac{1}{k} + \frac{1}{p} + \frac{k \cdot p}{2kp|k-p|} - \frac{2k^2 + kp + 2p^2}{kp|k-p|} \right. \\ &\quad \left. + \frac{2k^4 + 5k^2p^2 + 2p^4}{2(k \cdot p)kp|k-p|} \right] \\ &= \frac{1}{k} + \frac{1}{p} + \frac{kp}{6 \max(k^3, p^3)} - \frac{2k^2 + kp + 2p^2}{kp \max(k,p)} \\ &\quad + \frac{2k^4 + 5k^2p^2 + 2p^4}{2k^2p^2\sqrt{p^2+k^2}} \sinh^{-1} \frac{\min(p,k)}{\max(p,k)}\end{aligned}\quad (25)$$

where we have made use of the integrals given in Appendix A.

Thus, we arrive at the following nonlinear equation for the vacuum polarization:

$$\Pi(p) = \frac{Ne^2}{8p} - \frac{Ne^4}{4\pi^2 p} \int_0^\infty dk \frac{K(p,k)}{p[1+\Pi(k)]}. \quad (26)$$

Apparently, this equation is gauge invariant. We can rewrite it also as the equation for the running coupling $\bar{\alpha}(p)$ which must be self-consistently determined from it:

$$\bar{\alpha}^{-1}(p) = \bar{\alpha}_1^{-1}(p) - \frac{N}{4\pi^2 p} \int_0^\infty dk k K(p,k) \bar{\alpha}(k), \quad (27)$$

where $\bar{\alpha}_1(p)$ is the one-loop running coupling [see Eq. (1)].

Equation (27) is the simplest nonlinear equation for the running coupling (or the photon propagator) which is derived at the lowest order in the $1/N$ truncation of the SD equations. In fact, it should be considered as an analogue of the ladder approximation for the fermion propagator. The effects of a constant fermion mass can be incorporated at one’s wish by computing the box diagrams with massive fermions. This would allow one to study the coupled system of the SDE for the fermion self-energy and photon polarization operator along the lines of Ref. [18]. However, this is beyond the scope of the present paper and we shall leave aside this issue.³

Now we proceed by solving Eqs. (26) and (27) both analytically and numerically. Approximating, as usual, the expression (25) for the kernel by its asymptotics at $p \gg k$ and $p \ll k$

$$K(p,k) \simeq -\frac{2}{15} \frac{p^3 k^3}{\max(p^7, k^7)}, \quad (28)$$

one can reduce the integral Eqs. (26) and (27) to differential ones in order to study the asymptotical behavior of $\Pi(p)$ and $\bar{\alpha}(p)$ in the ultraviolet and infrared regions. However, in the present case we can find corresponding asymptotics directly from the integral equations.

First of all, we can immediately see that the solution of Eq. (27) for the running coupling possesses a nontrivial IR fixed point. Indeed, by making a change of variables, $k \rightarrow kp$, in the integral and assuming that $\bar{\alpha}(0) \neq 0$ we come to the quadratic equation for $\bar{\alpha}(0)$:

$$\bar{\alpha}^{-1}(0) = \bar{\alpha}_1^{-1}(0) - \frac{N}{4\pi^2} \int_0^\infty dk k K(1,k) \bar{\alpha}(0), \quad (29)$$

where we have made use of the fact that $pK(p,kp) = K(1,k)$, see Eq. (25). The last integral can be evaluated exactly (see Appendix A), and we obtain

$$\bar{\alpha}(0) = \frac{8}{NC},$$

$$\begin{aligned}C &= \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4}{N} \left(\frac{184}{9\pi^2} - 2 \right)} \simeq 1 + \frac{1}{N} \left(\frac{184}{9\pi^2} - 2 \right) \\ &\simeq 1 + \frac{1}{14.0N}.\end{aligned}\quad (30)$$

³A coupled system of SD equations for the vacuum polarization and the fermion renormalization wave function was studied in Ref. [15] using an ansatz for the full vertex satisfying the Ward-Takahashi (WT) identity. Though such an approach reproduces standard value for the critical $N_{\text{cr}} \approx 3.3$, it does not permit us to identify the ansatz with a class of Feynman diagrams.

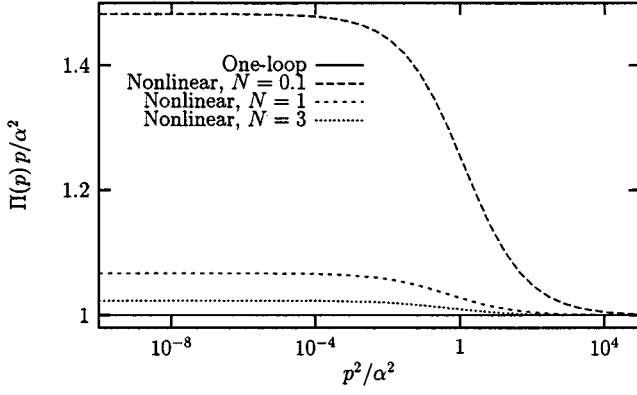


FIG. 6. Numerical solutions of Eq. (26).

This result illustrates that the $1/N$ expansion is reliable even for a rather low number of flavors, e.g., $N=2$, because of the smallness of the numerical coefficient in front of the $1/N$ term.

The next term in the expansion of $\bar{\alpha}(p)$ at small p can also be calculated exactly, as well as its asymptotics at large momenta but we focus on finding the asymptotics of the vacuum polarization operator itself. For it we seek a power solution $[\sim (p/\alpha)^\gamma]$ in both asymptotic regions, ($p \ll \alpha$) and ($p \gg \alpha$). We find that the power exponent can only be $\gamma = -1$ in both cases. Thus we get

$$\Pi(p) = C \frac{\alpha}{p}, \quad \text{for } p \ll \alpha, \quad (31)$$

$$\Pi(p) = \frac{\alpha}{p}, \quad \text{for } p \gg \alpha \quad (32)$$

with the constant C defined in Eq. (30) (we recall that $\alpha = e^2 N/8$). Hence for the running coupling we have

$$\bar{\alpha}(p) = \frac{e^2}{p(1+C\alpha/p)} \approx \frac{8}{CN}, \quad p \ll \alpha, \quad (33)$$

$$\bar{\alpha}(p) = \frac{e^2}{p(1+\alpha/p)}, \quad p \gg \alpha. \quad (34)$$

The numerical solution of Eq. (26) is presented in Fig. 6. From this figure it is clear that the IR behavior (i.e., $p \ll \alpha$) of $p\Pi(p)$ is indeed constant and in agreement with the analytic analysis.

For studying effects like symmetry breaking and dynamical mass generation, it is sufficient to consider only momenta less than α . Therefore for the remainder of this article we will just use Eq. (31) and treat α as the ultraviolet cutoff for nonperturbative dynamics. This allows us to write the gauge boson propagator as (in Euclidean formulation)

$$e^2 D_{\mu\nu}(p) = \left(-g_{\mu\nu} + [1 - \xi(p)] \frac{p_\mu p_\nu}{p^2} \right) \frac{\bar{\alpha}(0)}{p}, \quad \bar{\alpha}(0) = \frac{8}{NC}, \quad (35)$$

for $p = \sqrt{p^2} \ll \alpha$, with C given by Eq. (30), and where $\xi(p)$ parameterizes a nonlocal gauge fixing function (see Appendix B). This form of the photon propagator will be used in the next section.

The gauge boson propagator of Eq. (35) gives rise to a Coulomb potential instead of a logarithmically confining potential. The dimensionless coupling $\bar{\alpha}_0 \equiv \bar{\alpha}(0)$ should now be interpreted as the coupling parameter of a perfectly marginal (or conformal invariant) interaction: $\beta(\bar{\alpha}_0) = 0$.

III. QED₃ PLUS FOUR-FERMION INTERACTIONS

The gauged NJL model with N fermion flavors is described by the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{GNJL}} = & -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi \\ & + \frac{G}{2N} [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2], \end{aligned} \quad (36)$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative, and the last term is a chirally invariant four-fermion interaction with G the corresponding Fermi coupling constant. In the absence of a fermion mass term m_0 which breaks the chiral symmetry explicitly, the Lagrangian (36) possesses a $U(1)$ gauge symmetry and a global $U_L(1) \times U_R(1)$ chiral symmetry. For the four-fermion coupling we introduce the dimensionless coupling constant $g = 2G\Lambda/\pi^2$, and we consider the dimensionful gauge coupling e^2 as the UV cutoff (more precisely, $\alpha \simeq \Lambda$).

A parity invariant bare mass term $m_0 \bar{\psi}\psi$ as well as a dynamically generated fermion mass breaks the global symmetry down to $U_{L+R}(1)$. Further we study mainly the chiral symmetric case with $m_0 = 0$. By introducing the auxiliary scalar fields σ and π , the Lagrangian (36) can be rewritten as

$$\begin{aligned} \mathcal{L}_2 = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}i\gamma^\mu D_\mu\psi - \bar{\psi}(\sigma + i\gamma_5\pi)\psi \\ & - \frac{N}{2G}(\sigma^2 + \pi^2), \end{aligned} \quad (37)$$

where $\sigma = -(G/N)\bar{\psi}_i\psi_i$, $\pi = -(G/N)\bar{\psi}_i\gamma_5\psi_i$.

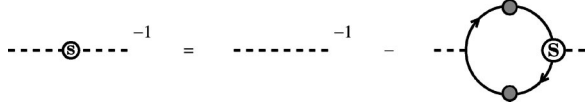
The propagators of the σ and π fields, Δ_S and Δ_P , are defined, respectively, as follows:

$$\Delta_S(q) = -i \int d^3x e^{iqx} \langle 0 | T[\sigma(x)\sigma(0)] | 0 \rangle_C, \quad (38)$$

$$\Delta_P(q) = -i \int d^3x e^{iqx} \langle 0 | T[\pi(x)\pi(0)] | 0 \rangle_C, \quad (39)$$

where the subscript C stands for ‘‘connected.’’ The SD equation for the scalar (pseudoscalar) propagator is given by

$$\Delta_{S(P)}^{-1}(p) = -\frac{N}{G} + \Pi_{S(P)}(p), \quad (40)$$

FIG. 7. The SDE for the scalar propagator $\Delta_S(p)$.

where the (pseudo)scalar vacuum polarization is

$$\Pi_{S(p)}(p) = i \int^\Lambda \frac{d^3k}{(2\pi)^3} \text{Tr}[S(k+p) \times \Gamma_{S(p)}(k+p, k) S(k) \Gamma_{0S(p)}] \quad (41)$$

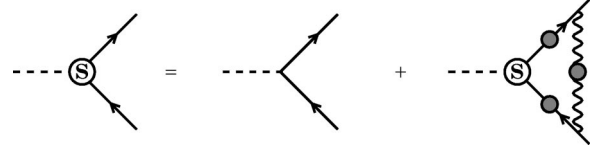
(see Fig. 7), $S(k)$ is the full fermion propagator [$S^{-1}(k) = \hat{k}A(k) - B(k)$], and $\Gamma_{S(p)}(k+p, k)$ is the fermion-antifermion (Yukawa) vertex (the bare Yukawa vertices are given by $\Gamma_{0S} = \mathbf{1}$, $\Gamma_{0P} = i\gamma_5$, where $\mathbf{1}$ is the identity matrix). The absence of kinetic terms for the σ and π fields in the Lagrangian is reflected in the constant bare propagator $-G$. The Yukawa vertices Γ_S and Γ_P are defined as the “fully amputated” vertices,

$$S(k)\Gamma_S(k, p)S(p)\Delta_S(k-p) = - \int d^3x d^3y e^{ikx - ipy} \langle 0 | T[\psi(x) \bar{\psi}(y) \sigma(0)] | 0 \rangle_C, \quad (42)$$

$$S(k)\Gamma_P(k, p)S(p)\Delta_P(k-p) = - \int d^3x d^3y e^{ikx - ipy} \langle 0 | T[\psi(x) \bar{\psi}(y) \pi(0)] | 0 \rangle_C. \quad (43)$$

In the symmetric phase of the GNJL model the pseudoscalar and scalar propagators are degenerate, so are the pseudoscalar vertex and scalar vertex.

We shall study the SDE for the Yukawa vertex Γ_S and scalar propagator Δ_S with both the gauge interaction and the four-fermion interactions treated in the leading order of the $1/N$ expansion. This approximation is obtained by replacing the Bethe-Salpeter kernel K by planar one photon exchange graph with the photon propagator given by Eq. (35) and bare fermion-photon vertices (see Fig. 2). In principle the Bethe-Salpeter kernel also contains scalar and pseudoscalar exchanges. One can question whether such exchanges can be neglected. In fact, if one includes the ladder like one-scalar and one-pseudoscalar exchanges in the truncation of the BS kernel K in the SDE for the Yukawa vertices, then such contributions cancel each other exactly in the symmetric phase. On the other hand, in the equation for the fermion wave function $A(p)$ these contributions add and must be taken into account. Since we take the bare vertex approximation, we need to set $A(p) = 1$ for consistency with the WT identity. In Appendix B we prove the existence of such a nonlocal gauge for the GNJL model in the bare vertex ap-

FIG. 8. The SDE for the scalar Yukawa vertex Γ_S in the ladder approximation.

proximation and in arbitrary dimensions.⁴ There it is shown also that four-fermion contributions into the gauge function ξ are suppressed leading to $\xi(p) = 2/3$ (Nash’s nonlocal gauge). In what follows we use the Nash gauge for the photon propagator (35).

The equation for the Yukawa vertex, within the proposed approximation, reads

$$\Gamma_S(p+q, p) = \mathbf{1} + ie^2 \int^\Lambda \frac{d^3k}{(2\pi)^3} \gamma^\lambda S(k+q) \times \Gamma_S(k+q, k) S(k) \gamma^\sigma D_{\lambda\sigma}(k-p) \quad (44)$$

(see Fig. 8). In the symmetric phase, the equation for the scalar vertex, Eq. (44), is a self-consistent equation if one uses a gauge where the full fermion propagator has the form of the free or bare fermion propagator $S(p) = S_0(p) = 1/\hat{p}$.

The invariance under parity and charge conjugation restricts the form of the Yukawa vertices to the following decomposition [30,24]:

$$\Gamma_S(p+q, p) = \mathbf{1}[F_1(p+q, p) + (\hat{q}\hat{p} - \hat{p}\hat{q})F_2(p+q, p)], \quad (45)$$

$$\Gamma_P(p+q, p) = (i\gamma_5)[F_1(p+q, p) + (\hat{q}\hat{p} - \hat{p}\hat{q})F_2(p+q, p)] \quad (46)$$

in the symmetric phase. The two scalar functions F_i are symmetric in the fermion momenta:

$$F_i(p+q, p) \equiv F_i[(p+q)^2, p^2, q^2] = F_i[p^2, (p+q)^2, q^2], \quad i = 1, 2. \quad (47)$$

This is analogous to the four-dimensional case.

Since we are considering the symmetric phase, the σ and π propagators are identical. In what follows we neglect the contribution of F_2 to the Yukawa vertices. The validity of this approximation was argued in Ref. [24] for the four-dimensional case and the analysis can be generalized straightforwardly to the three-dimensional case. Here we only point out that calculating F_1, F_2 in $1/N$ perturbation theory reveals that the function F_1 contains logarithmic terms which build up into the power-law form of the full solution (see below); on the other hand, F_2 does not contain such logarithmic terms and thus will not contribute to the leading and next-to-leading in $1/N$ order behavior of Π_S .

⁴A version of a nonlocal gauge in $D=4$ leading to approximate equality $A = 1$ was proposed in Ref. [29].

Hence, neglecting all functions except F_1 , we obtain (in Euclidean formulation) after substituting Eq. (35) with $\xi(p) = 2/3$ in Eq. (44)

$$F_1(p+q, p) = 1 + \lambda \int_0^\Lambda dk \int \frac{d\Omega}{4\pi} \frac{(k^2 + q \cdot k)}{(k+q)^2} \frac{1}{|k-p|} \times F_1(k+q, k), \quad (48)$$

where $\lambda = 32/(3NC\pi^2)$ and where $\int d\Omega$ denotes the usual angular part of the three-dimensional integration. The equation for the σ boson vacuum polarization is

$$\Pi_S(q) = \frac{2N}{\pi^2} \int_0^\Lambda dk \int \frac{d\Omega}{4\pi} \frac{(k^2 + q \cdot k)}{(k+q)^2} F_1(k+q, k). \quad (49)$$

To resolve the angular dependence of the Yukawa vertex function F_1 it is convenient to use an expansion in Legendre polynomials P_n (see also Appendix C),

$$F_1(p+q, p) = F_1(p, p+q) = \sum_{n=0}^{\infty} f_n(p, q) P_n(p \cdot q / pq), \quad (50)$$

where in the right-hand side expression $p = \sqrt{p^2}$, $q = \sqrt{q^2}$, and $p \cdot q / pq = \cos \alpha$. Then we follow the arguments of Ref. [24] and assume that the Yukawa vertex function $F_1(p+q, p)$ depends only weakly on the angle $p \cdot q / pq$ between fermion and σ boson momenta, so that the set equations for f_n reduces to the equation for the zeroth-order Legendre coefficient function f_0 only. This is equivalent to approximating Γ_S by its angular average

$$\Gamma_S(p+q, p) = \Gamma_S(p, p+q) \approx \mathbf{1} \int \frac{d\Omega}{4\pi} F_1(p+q, p) = \mathbf{1} f_0(p, q), \quad (51)$$

where the function $f_0(p, q)$ depends on the absolute values of the vectors p, q , i.e., in it $p = \sqrt{p^2}$, $q = \sqrt{q^2}$. Accordingly we write

$$f_0(p, q) = F_{\text{IR}}(p, q) \theta(q-p) + F_{\text{UV}}(p, q) \theta(p-q), \quad (52)$$

where the functions F_{IR} and F_{UV} satisfy integral equations which are given in Appendix C [see Eqs. (C16) and (C17)]. Within this approximation, we find that the scalar vacuum polarization (C9) is expressed through the function F_{UV} [see Eqs. (C9), (C11), and (C13)]:

$$\Pi_S(q) = \frac{2\Lambda}{\pi^2} \frac{N}{\lambda} [F_{\text{UV}}(\Lambda, q) - 1]. \quad (53)$$

The integral Eqs. (C16) and (C17) can be reduced to second order differential equations

$$p^2 \frac{d^2}{dp^2} F_{\text{IR}} + 2p \frac{d}{dp} F_{\text{IR}} + \lambda \frac{p^2}{2q^2} F_{\text{IR}} = 0, \quad (54)$$

$$p^2 \frac{d^2}{dp^2} F_{\text{UV}} + 2p \frac{d}{dp} F_{\text{UV}} + \lambda \left(1 - \frac{q^2}{2p^2}\right) F_{\text{UV}} = 0, \quad (55)$$

with four boundary conditions. The infrared and ultraviolet boundary conditions (IRBC and UVBC), respectively, are

$$\left[p^2 \frac{d}{dp} F_{\text{IR}}(p, q) \right]_{p=0} = 0, \quad \left[F_{\text{UV}}(p, q) + p \frac{d}{dp} F_{\text{UV}}(p, q) \right]_{p=\Lambda} = 1. \quad (56)$$

There is a continuity and differentiability equation at $p=q$:

$$F_{\text{IR}}(q, q) = F_{\text{UV}}(q, q), \quad \left. \frac{d}{dp} F_{\text{IR}}(p, q) \right|_{p=q} = \left. \frac{d}{dp} F_{\text{UV}}(p, q) \right|_{p=q}. \quad (57)$$

The equation for F_{UV} can be written as

$$z^2 \frac{d^2}{dz^2} F_{\text{UV}} + (\lambda - z^2) F_{\text{UV}} = 0, \quad z = \sqrt{\frac{\lambda}{2}} \frac{q}{p}. \quad (58)$$

The differential Eqs. (54) and (58) and the BC's (56) and (57) can be solved straightforwardly. The solutions are

$$F_{\text{IR}}(p, q) = Z^{-1} \left(\frac{q}{\Lambda}, \omega \right) \left(\frac{q}{p} \right) \sin \left(\sqrt{\frac{\lambda}{2}} \frac{p}{q} \right), \quad (59)$$

$$F_{\text{UV}}(p, q) = \frac{\pi}{2 \sin(\omega \pi / 2)} Z^{-1} \left(\frac{q}{\Lambda}, \omega \right) \times \left(\frac{q}{p} \right)^{1/2} \left[\rho(\omega) I_{-\omega/2} \left(\sqrt{\frac{\lambda}{2}} \frac{q}{p} \right) - \rho(-\omega) I_{\omega/2} \left(\sqrt{\frac{\lambda}{2}} \frac{q}{p} \right) \right], \quad (60)$$

where $I_{\pm\nu}$ are modified Bessel functions, and ω is given by

$$\omega = \sqrt{1 - 4\lambda} = \sqrt{1 - N_{\text{cr}}/N}, \quad N_{\text{cr}} = 128/(3C\pi^2). \quad (61)$$

Furthermore,

$$Z(q/\Lambda, \omega) \equiv \frac{\pi}{2 \sin(\omega \pi / 2)} [\rho(\omega) R(q/\Lambda, -\omega) - \rho(-\omega) R(q/\Lambda, \omega)], \quad (62)$$

and

$$\rho(\omega) \equiv I_{\omega/2} \left(\sqrt{\frac{\lambda}{2}} \right) \left[\sqrt{\frac{\lambda}{2}} \cos \sqrt{\frac{\lambda}{2}} - \frac{1}{2} \sin \sqrt{\frac{\lambda}{2}} \right] + I'_{\omega/2} \left(\sqrt{\frac{\lambda}{2}} \right) \left[\sqrt{\frac{\lambda}{2}} \sin \sqrt{\frac{\lambda}{2}} \right], \quad (63)$$

$$R(q/\Lambda, \omega) \equiv \frac{1}{2} \sqrt{\frac{q}{\Lambda}} \left[I_{\omega/2} \left(\sqrt{\frac{\lambda}{2}} \frac{q}{\Lambda} \right) - 2 \sqrt{\frac{\lambda}{2}} \frac{q}{\Lambda} I'_{\omega/2} \left(\sqrt{\frac{\lambda}{2}} \frac{q}{\Lambda} \right) \right]. \quad (64)$$

The $\sin \omega\pi/2$ results from the Wronskian between $I_{-\omega/2}(x)$ and $I_{\omega/2}(x)$.

By adopting the approximation (51) we have obtained an analytic expression for the Yukawa vertex Γ_S . Within this approximation, the σ boson propagator Δ_S defined by Eq. (40) is related to Γ_S via Eq. (53). Such an expression is valid in the symmetric phase of the phase diagram.

IV. SCALING AND OTHER PROPERTIES

In the previous section we have obtained nonperturbative solutions for the Yukawa vertex and scalar propagator within the ladder approximation. In this section we discuss some important properties of the Yukawa vertex and scalar propagator.

Let us briefly state our objectives. First, we apply the Thouless criterion of the symmetry phase instability in order to derive the critical curve given in Eq. (3). Subsequently, we show that near this curve the scalar propagator has a scaling form consistent with the general renormalization group theory of second order phase transitions. We find that the anomalous dimension of the propagator of the composite scalar fields is $\eta = 2 - \omega$ with ω given by Eq. (61). Moreover, we show that the Yukawa vertex has a scaling form consistent with power-law renormalizability. Second, we derive the peculiar behavior of Π_S near $N = N_{\text{cr}}$. The phase transition at $N = N_{\text{cr}}$ is known as the CPT and is characterized by the absence of light unstable modes in the symmetric phase. Another characteristic feature of the CPT is the scaling law with essential singularity for the scalar boson and the fermion mass in the broken phase. This scaling law can be obtained by analytical continuation of Π_S in ω across the critical curve at $N = N_{\text{cr}}$.

In analogy with Ref. [24] we investigate a few specific limits:

(i) the large flavor limit ($N \rightarrow \infty$), which means that the gauge interaction is negligible with respect to four-fermion interactions, i.e., $\lambda = 0$, thus $\omega = 1$; (ii) asymptotic or IR behavior of $\Gamma_S(p+q, p)$ and $\Delta_S(q)$, i.e., $p, q \ll \Lambda$; (iii) the behavior of Π_S at the critical coupling $\lambda = \lambda_c = 1/4$, thus $\omega = 0$; (iv) the behavior of Π_S for $\lambda > \lambda_c$, $\omega = i\nu$, $\nu = \sqrt{4\lambda - 1}$, i.e., analytic continuation across the critical curve at $\lambda = \lambda_c$.

A. Large flavor limit

In the large flavor limit, the four-fermion interactions completely govern the dynamical breakdown of ‘‘chiral’’ symmetry. In this limit $\omega = 1$ ($\lambda = 0$), thus the Yukawa vertex (44) is $\Gamma_S(p+q, p) = 1$. Consequently, we obtain an expression for Π_S from Eq. (C9) by using Eq. (C13) and $F_{\text{UV}}(p, q) = F_{\text{IR}}(p, q) = 1$ at $\omega = 1$. This leads to

$$\Pi_S(q) = \frac{2N\Lambda}{\pi^2} \left[1 - \frac{4q}{3\Lambda} + \frac{q^2}{2\Lambda^2} \right]. \quad (65)$$

This expression is obtained by making use of the approximation (C13). Naturally, the expression for $\Pi_S(q)$ can be obtained by evaluating Eq. (41) with $\Gamma_S = 1$. The result is

$$\Pi_S(q) = \frac{2N\Lambda}{\pi^2} \left[1 - \frac{\pi^2}{8} \frac{q}{\Lambda} + \frac{q^2}{3\Lambda^2} \right], \quad (66)$$

see, e.g., Ref. [32] and references therein [33]. Since only the first two terms on the right-hand side of Eqs. (65) and (66) are important in the IR ($q \ll \Lambda$), these equations differ about 10%.

B. Asymptotic behavior and scaling

For values $0 < \omega < 1$, the asymptotic behavior or IR behavior of Γ_S and Π_S with $(q/\Lambda)^\omega \gg q/\Lambda$ can be derived by first considering the $q \ll \Lambda$ limit of Z , Eq. (62):

$$Z \approx \frac{\pi}{2 \sin(\omega\pi/2)} \left(\frac{q}{\Lambda} \right)^{1/2} C(\omega) \sinh \left[\frac{\omega}{2} \ln \frac{\Lambda}{q} + \delta(\omega) \right], \quad (67)$$

where

$$\delta(\omega) = \frac{1}{2} \ln \frac{\rho(\omega)(1+\omega)\Gamma(1+\omega/2)}{\rho(-\omega)(1-\omega)\Gamma(1-\omega/2)} - \frac{\omega}{4} \ln \frac{\lambda}{8}, \quad (68)$$

$$C(\omega) = \sqrt{\frac{\rho(\omega)\rho(-\omega)(1-\omega^2)}{\Gamma(1+\omega/2)\Gamma(1-\omega/2)}}. \quad (69)$$

In this limit, the function $F_{\text{UV}}(p, q)$ with fermion momentum $p = \Lambda$ can be expressed as

$$F_{\text{UV}}(\Lambda, q) \approx \frac{2}{1+\omega} + \frac{2\omega}{(1-\omega^2)} (1 - \coth y),$$

$$y = \frac{\omega}{2} \ln \frac{\Lambda}{q} + \delta(\omega). \quad (70)$$

Thus, by using Eq. (53), the asymptotic form for Π_S reads

$$\Pi_S(q) \approx \frac{2N\Lambda}{\pi^2} \left[\frac{4}{(1+\omega)^2} + \frac{8\omega}{(1-\omega^2)^2} (1 - \coth y) \right]. \quad (71)$$

Hence

$$\Pi_S(q) \approx \frac{2N\Lambda}{\pi^2} \left[\frac{1}{g_c} - B(\omega) \left(\frac{q}{\Lambda} \right)^\omega + \mathcal{O}[(q/\Lambda)^{2\omega}] + \mathcal{O}[(q/\Lambda)^2] \right], \quad q \ll \Lambda, \quad (72)$$

where

$$g_c = \frac{(1+\omega)^2}{4},$$

$$B(\omega) \equiv \frac{16\omega}{(1+\omega)^3(1-\omega)} \frac{\rho(-\omega)}{\rho(\omega)} \frac{\Gamma\left(1-\frac{\omega}{2}\right)}{\Gamma\left(1+\frac{\omega}{2}\right)} \left(\sqrt{\frac{\lambda}{8}}\right)^\omega. \quad (73)$$

One can show that $B(1)=4/3$, which is in agreement with Eq. (65). The expression (72) for the asymptotic behavior of $\Pi_S(q)$ is valid for $0 < \omega \leq 1$, but not for $\omega=0$ ($\lambda=\lambda_c$).

The inverse propagator Δ_S^{-1} that follows from Eqs. (40) and (72) is given by

$$\Delta_S^{-1}(q) \approx -\frac{2B(\omega)N\Lambda}{\pi^2} \left[\frac{1}{B(\omega)} \left(\frac{1}{g} - \frac{1}{g_c} \right) + \left(\frac{q}{\Lambda} \right)^\omega \right]. \quad (74)$$

The instability of the symmetric phase is signaled by the vanishing of $\Delta_S^{-1}(q=0)$. This is nothing else than the Thouless criterion for a phase transition of the second kind [34] which leads to the critical curve

$$g = g_c, \quad 0 < \omega < 1 \quad (N > N_{cr}), \quad g > \frac{1}{4}. \quad (75)$$

Thus the curve $g=g_c$ is a line of UV stable fixed points. On the critical line the scalar propagator scales as

$$\Delta_S(q) \approx -\frac{\pi^2}{2B(\omega)N\Lambda} \left(\frac{\Lambda}{q} \right)^{2-\eta}, \quad \eta = 2 - \omega, \quad (76)$$

where η is the anomalous dimension.

On the other hand, one can see that on the line $\omega=0$ ($N=N_{cr}$), $g < 1/4$, $\Delta_S^{-1}(q=0)$ does not vanish. Nevertheless, as we shall show in Sec. V, this line is also the phase transition line but of a special type.

The scaling form for Γ_S is obtained by considering only the leading term in Eq. (67). Thus the Z function scales as

$$Z(q/\Lambda, \omega) \approx \frac{\pi}{2 \sin(\omega\pi/2)} \frac{\rho(\omega)}{2} \times \frac{(1+\omega)}{\Gamma(1-\omega/2)} \left(\frac{\lambda}{8} \right)^{-\omega/4} \left(\frac{q}{\Lambda} \right)^{(1-\omega)/2}. \quad (77)$$

In this way the Yukawa vertex can be written as

$$\Gamma_S(p+q, p) \approx \mathbf{1} \left(\frac{\Lambda}{q} \right)^{(\eta-1)/2} [\mathcal{F}_{\text{IR}}(p/q) \theta(q-p) + \mathcal{F}_{\text{UV}}(q/p) \theta(p-q)], \quad (78)$$

where, for $p, q \ll \Lambda$,

$$F_{\text{IR}}(p, q) \approx \left(\frac{\Lambda}{q} \right)^{(\eta-1)/2} \mathcal{F}_{\text{IR}}(p/q),$$

$$F_{\text{UV}}(p, q) \approx \left(\frac{\Lambda}{q} \right)^{(\eta-1)/2} \mathcal{F}_{\text{UV}}(q/p), \quad (79)$$

and

$$\mathcal{F}_{\text{IR}}(p/q) = \frac{2 \sin(\omega\pi/2)}{\pi} \frac{2}{\rho(\omega)} \frac{\Gamma(1-\omega/2)}{(1+\omega)} \left(\frac{\lambda}{8} \right)^{\omega/4} \left(\frac{q}{p} \right) \times \sin \left(\sqrt{\frac{\lambda p}{2q}} \right), \quad (80)$$

$$\mathcal{F}_{\text{UV}}(q/p) = \frac{2}{\rho(\omega)} \frac{\Gamma(1-\omega/2)}{(1+\omega)} \left(\frac{\lambda}{8} \right)^{\omega/4} \left(\frac{q}{p} \right)^{1/2} \times \left[\rho(\omega) I_{-\omega/2} \left(\sqrt{\frac{\lambda q}{2p}} \right) - \rho(-\omega) I_{\omega/2} \left(\sqrt{\frac{\lambda q}{2p}} \right) \right]. \quad (81)$$

An important consequence of the scaling behavior of the scalar propagator [Eq. (76)] and of the Yukawa vertex [Eq. (78)] is that, in using them, one finds that the four-fermion scattering amplitudes scale as

$$\Gamma_S(p_1+q, p_1) \Delta_S(q) \Gamma_S(p_2, p_2+q) \propto \frac{1}{q}, \quad p_1, p_2 \ll q \ll \Lambda. \quad (82)$$

This scaling form reveals the long range nature and power-law renormalizability of the four-fermion interactions at the phase transition line [35].

C. At the critical coupling

At the critical value of λ , i.e., $\omega=0$ ($\lambda_c=1/4$), we can derive in analogy with Ref. [24] that for $p \gg q$

$$F_{\text{UV}}(p, q) \approx 2 \left(\frac{p}{\Lambda} \right)^{-1/2} \left[\frac{\epsilon_3 - 2 + \ln(p/q)}{\epsilon_3 - \ln(q/\Lambda)} + \mathcal{O}[q^2/p^2 \ln(q/p)] \right], \quad (83)$$

where

$$\epsilon_1 = I_0(\sqrt{1/8}) \left[\sqrt{1/8} \cos \sqrt{1/8} - \frac{1}{2} \sin \sqrt{1/8} \right] + I_0'(\sqrt{1/8}) \left[\sqrt{1/8} \sin \sqrt{1/8} \right], \quad (84)$$

$$\epsilon_2 = K_0(\sqrt{1/8}) \left[\sqrt{1/8} \cos \sqrt{1/8} - \frac{1}{2} \sin \sqrt{1/8} \right] + K_0'(\sqrt{1/8}) \left[\sqrt{1/8} \sin \sqrt{1/8} \right], \quad (85)$$

$$\epsilon_3 = 2 - \gamma + \frac{5}{2} \ln 2 - \frac{\epsilon_2}{\epsilon_1}, \quad (86)$$

with γ the Euler gamma and K_0 the modified Bessel function of the third kind.

In the infrared, i.e., $q \ll \Lambda$, Π_S can be written as

$$\Pi_S(q) \approx \frac{2N\Lambda}{\pi^2} \left[4 + \frac{16}{\ln(q/\Lambda) - \epsilon_3} + \mathcal{O}[q^2/\Lambda^2 \ln(q/\Lambda)] \right]. \quad (87)$$

This straightforwardly follows from the insertion of Eq. (83) in Eq. (53).

D. Analytic continuation across the critical curve

Since the expression for the σ boson vacuum polarization is symmetric under replacement of ω by $-\omega$, it can be analytically continued to the values $\lambda > \lambda_c$. This holds in replacing ω by $i\nu$ in Eq. (53) with F_{UV} given by Eq. (60), where

$$\nu = \sqrt{4\lambda - 1}. \quad (88)$$

In the infrared ($q \ll \Lambda$), it means that Π_S can be written as

$$\Pi_S(q) \approx \frac{2N\Lambda}{\pi^2} \left[\frac{4(1-\nu^2)}{(1+\nu^2)^2} - \frac{8\nu}{(1+\nu^2)^2} \cot y \right], \quad (89)$$

$$y = \frac{\nu}{2} \ln \frac{\Lambda}{q} + \nu \phi(\nu^2),$$

where we have used Eq. (71) with $\omega \rightarrow i\nu$, and where $\phi(\nu^2) = \delta(i\nu)/i\nu$.

The four limits of Π_S described above are very useful for illustrating the resonance structure of the bound states and peculiar dynamics of the CPT, see Sec. V.

To conclude this section let us mention that at zero σ boson momentum ($q=0$), we obtain

$$\Gamma_S(p, p) = F_{UV}(p, q=0) = \frac{2}{1+\omega} \left(\frac{p}{\Lambda} \right)^{-(1-\omega)/2}. \quad (90)$$

V. LIGHT RESONANCES AND THE CONFORMAL PHASE TRANSITION

In this section we analyze the behavior of the σ boson propagator near the critical line in the symmetric phase ($g \leq g_c$), where the σ and π boson are degenerate. We will show that for $\omega > 0$ ($N > N_{cr}$) the scalar composites (σ and π bosons) are resonances (unstable modes) described by a complex pole in their respective propagators. The complex pole in Δ_S should lie on a second or higher Riemann sheet [i.e., not on the first (physical) sheet] of the complex plane of the Minkowskian momentum p^2 , because unitarity (causality) demands that $\Delta_S(p)$ is analytic in the upper half of the complex p_0 -plane, where p_0 is the ‘‘time’’ component of the Minkowski momentum $p^2 = p_0^2 - \vec{p}^2$.

From Eq. (74) the complex pole can be computed. First we rotate back to Minkowski space, $p^2 \rightarrow p_M^2 \exp(-i\pi)$. Sub-

sequently, the complex poles are given by

$$p_M^2 = |m_\sigma|^2 \exp(-i\theta), \quad \Delta_S^{-1}(p_M) = -\frac{2N\Lambda}{\pi^2 g} + \Pi_S(p_M) = 0. \quad (91)$$

The equation for the imaginary part reads

$$0 \approx \sin \frac{\omega(\theta + \pi)}{2}, \quad (92)$$

with the solution

$$\theta \approx -\pi + \frac{2n\pi}{\omega}, \quad (93)$$

where n is an odd integer. Hence for values $0 < \omega \leq 1$ it follows from Eq. (93) that the complex pole does not lie on the physical sheet of p^2 .⁵ Since $\cos \omega(\theta + \pi)/2 = -1$, we find that the solution for $|m_\sigma|$ is

$$\frac{|m_\sigma|}{\Lambda} = \left[\frac{\Delta g}{g_c g B(\omega)} \right]^{1/\omega}, \quad \Delta g = g_c - g, \quad (94)$$

consequently the critical exponent $\nu = 1/\omega$ [36].⁶ Equation (94) describes how the mass of the pole vanishes as g is tuned toward the critical line.

The propagator Δ_S is of the form given by Eq. (74) and in Minkowski space, with the definition $p = \sqrt{p^2}$, it can be written as follows:

$$\Delta_S(p) = \frac{\pi^2}{2N\Lambda} \frac{g g_c}{\Delta g} \left[\frac{-1}{1 + (-1)^{\omega/2} (p/|m_\sigma|)^\omega} \right], \quad (95)$$

with $|m_\sigma|$ given by Eq. (94). Then, the real and imaginary part of Δ_S are

$$\begin{aligned} & \text{Re}[\Delta_S(p)] \\ &= - \left(\frac{\pi^2}{2N\Lambda} \frac{g g_c}{\Delta g} \right) \frac{[1 + (p/|m_\sigma|)^\omega \cos \varphi]}{(p/|m_\sigma|)^{2\omega} + 2(p/|m_\sigma|)^\omega \cos \varphi + 1}, \end{aligned} \quad (96)$$

⁵We denote the first (physical) Riemann sheet of p^2 by angles θ with $0 \leq -\theta < 2\pi$ (the origin is a branch point with a branch cut along the positive real axis).

⁶The critical exponents ν and η coincide with those found in Ref. [36]. Note, however, that in Ref. [36] η was obtained assuming the validity of scaling relations between the critical exponents of the theory. Thus the independent computation of ν , η in the present paper gives, in fact, a proof of the scaling relations.

$$\begin{aligned} \text{Im}[\Delta_S(p)] = & -\left(\frac{\pi^2}{2N\Lambda} \frac{g g_c}{\Delta g}\right) \\ & \times \frac{(p/|m_\sigma|)^\omega \sin \varphi}{(p/|m_\sigma|)^{2\omega} + 2(p/|m_\sigma|)^\omega \cos \varphi + 1}, \end{aligned} \quad (97)$$

where $\varphi = \pi\omega/2$. The absolute value of the imaginary part has a maximum at $p = |m_\sigma|$ and the maximum is

$$\text{Im}[\Delta_S(|m_\sigma|)] = -\frac{\pi^2}{2N\Lambda} \frac{g g_c}{\Delta g} \frac{\sin \varphi}{2[\cos \varphi + 1]}. \quad (98)$$

This shows that when g approaches g_c from below ($g \uparrow g_c$), $|m_\sigma|$ goes to zero ($|m_\sigma| \rightarrow 0$) and that the maximum of the absolute value of the imaginary part of Δ_S approaches infinity [$-\text{Im} \Delta_S(|m_\sigma|) \rightarrow \infty$].

We define a width over mass ratio $\Gamma/|m_\sigma|$ as follows:

$$\frac{\Gamma}{|m_\sigma|} = \frac{p_+}{|m_\sigma|} - \frac{p_-}{|m_\sigma|}, \quad \text{Im}[\Delta_S(p_\pm)] = \frac{1}{2} \text{Im}[\Delta_S(|m_\sigma|)]. \quad (99)$$

Thus the width is the difference between the momenta at which $-\text{Im}(\Delta_S)$ equals 1/2 of the maximum value of $-\text{Im}(\Delta_S)$. Solving Eq. (99) by making use of Eqs. (97) and (98) gives

$$\begin{aligned} \frac{\Gamma}{|m_\sigma|} = & [2 + \cos \varphi + \sqrt{(2 + \cos \varphi)^2 - 1}]^{1/\omega} \\ & - [2 + \cos \varphi - \sqrt{(2 + \cos \varphi)^2 - 1}]^{1/\omega}. \end{aligned} \quad (100)$$

Thus, as the mass scale of the pole is made small by approaching the critical line, the resonance is not described by a narrow Breit-Wigner type, because the width over mass ratio is rather large. Consequently, the resonance does not have the Lorentzian shape which is a characteristic feature of the Breit-Wigner resonance [note that even in pure NJL model ($\omega=1$) the resonance is not narrow in contrast to four-dimensional NJL model]. The above expression shows also that $\Gamma/|m_\sigma|$ increases when $\omega \rightarrow 0$, and the resonance becomes broader.

A description of the resonance structure is provided by a plot of $\text{Im} \Delta_S(p)$. This is illustrated in Fig. 9 in which $\text{Im}[\Delta_S(p)]/\Delta_S(0)$ is drawn as a function of the energy scale $p/|m_\sigma|$ for various values of ω .

Absence of light resonances near N_{cr}

The existence of light resonances whose mass vanishes as the transition is approached from the side of symmetric phase in (2+1)-dimensional theories is relevant for describing spin excitations in high- T_c cuprate superconductors (see the paper by Kim and Lee [25] and references therein). Such resonances can be considered as precursors of the antiferromagnetic transition. It is known that QED₃ by itself cannot give rise to light excitations in the symmetric phase [37,5,6]. This is one of the main features of the so-called conformal

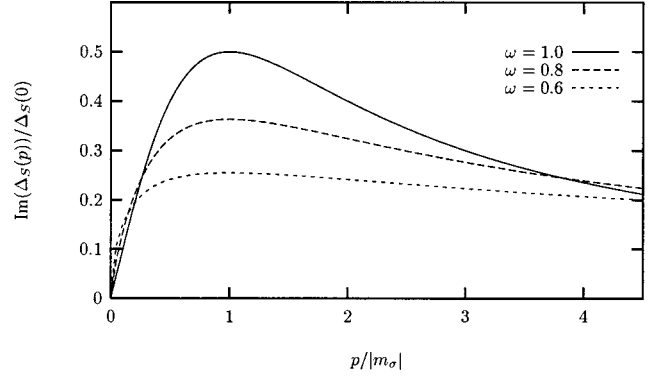


FIG. 9. The response function $\text{Im}[\Delta_S(p)]/\Delta_S(0)$ vs $p/|m_\sigma|$ for $\omega = 1.0$, $\omega = 0.8$, $\omega = 0.6$.

phase transition: the absence of light excitations (composites) in the symmetric phase as the transition is approached (in the broken phase massless ‘‘normal’’ Goldstone bosons appear). This unusual behavior can be attributed to the long range nature of the gauge interaction in the model under consideration. Another characteristic feature of the CPT is the scaling law with essential singularity for the dynamical fermion mass in the broken phase [5].

From the side of the symmetric phase there is no sign indicating the occurrence of a phase transition. This means that the correlation length remains finite in the symmetric phase even close to the critical point (the Thouless criterion is not valid). In QED₃ the CPT occurs at $\lambda = \lambda_c$ ($N = N_{\text{cr}}$) where the symmetry is dynamically broken by a ‘‘marginal’’ operator (a long range interaction). Though continuous, the CPT is not a second order phase transition. This is reflected by the singular behavior of some of the critical exponents (e.g., ν and β , see Ref. [36]) as ω goes to zero. The absence of a light complex pole in the σ boson propagator illustrates the CPT in GNJL model in 2+1 dimensions. At $\omega=0$ the σ boson vacuum polarization is given by Eq. (87) in the infrared. If there has to be a light excitation in the symmetric phase then there must be a complex pole $p_M^2 = |m_\sigma|^2 \exp(-i\theta)$ in Δ_S with $|m_\sigma| \ll \Lambda$ as $g < 1/4$. From Eq. (87), we then should find zeros of Δ_S^{-1} at

$$0 \approx \left(\frac{1}{g} - 4\right) + \frac{16[\ln(\Lambda/|m_\sigma|) + \epsilon_3]}{[\ln(\Lambda/|m_\sigma|) + \epsilon_3]^2 + (\theta + \pi)^2/4}, \quad (101)$$

$$0 \approx \theta + \pi. \quad (102)$$

For $g \leq g_c = 1/4$, there are no solutions satisfying $|m_\sigma| \ll \Lambda$, hence if there is a pole it will be heavy, i.e., $|m_\sigma| \sim \Lambda$. Therefore at $\lambda = \lambda_c$ and $g < 1/4$ there are no light resonances in the 2+1-dimensional GNJL model.

What happens with the σ -boson propagator if we analytically continue it to the values $\lambda > \lambda_c$, $N < N_{\text{cr}}$? By doing so, we remain in the massless chiral symmetric phase, but we just end up in the ‘‘wrong vacuum’’ (the chiral symmetric

solution becomes unstable).⁷ The π and σ bosons are tachyons for such a solution. Thus the border of the stable symmetric solution $\lambda = \lambda_c, g < 1/4$ is also the phase transition line.

Let us show that there are indeed tachyons (with imaginary mass $m^2 < 0$) when $\lambda > \lambda_c$. For this we need to show that Δ_S has a real pole in Euclidean momentum space. Assuming that the pole lies in the infrared, $|m_\sigma| \ll \Lambda$, we can use Eq. (89), where ω has been replaced by $i\nu$, ν given by Eq. (88). The tachyonic pole is given by

$$-\frac{2N\Lambda}{\pi^2 g} + \Pi_S(|m_\sigma|) = 0. \quad (103)$$

From this we derive the solution

$$\frac{|m_\sigma|}{\Lambda} = \exp\left(-\frac{2n\pi}{\nu} - \frac{2\beta}{\nu} + 2\phi(\nu^2)\right), \quad (104)$$

where

$$\beta = \tan^{-1} \frac{\nu g}{g - 2\lambda(g + \lambda)}. \quad (105)$$

As is well established, the tachyon with the largest $|m_\sigma|$ in the physical region corresponds to $n = 1$.

It is clear that the tachyons in the symmetric solution appear also when we cross the upper part ($\lambda < \lambda_c, g > 1/4$) of the critical curve. However, the difference between this part of the critical line and the line $\lambda = \lambda_c, g < 1/4$ is that we have light composites (resonances) near the first line while they are absent near the last one.

If we now consider the limit $\lambda \downarrow \lambda_c$, i.e., we approach the transition from the side of the broken phase, we obtain the scaling law with essential singularity,

$$\frac{|m_\sigma|}{\Lambda} \approx \exp\left(\frac{4g}{1/4 - g} + 2\phi(0)\right) \exp\left(-\frac{2\pi}{\sqrt{4\lambda - 1}}\right). \quad (106)$$

This scaling law with essential singularity is obtained by analytical continuation of the solution in the symmetric phase ($\lambda < \lambda_c$) to the broken phase ($\lambda > \lambda_c$). Thus the tachyonic (unphysical) solution in the broken phase leads to a scaling law which is proportional to the scaling law given by the fermion mass and σ boson mass in the broken phase [2].

VI. CONCLUSION

In this paper we studied a nonlinear equation for the running coupling in QED₃ which can be considered as the analogue of the ladder approximation for the fermion propagator. We solved our equation both analytically and numerically. We found that the vacuum polarization opera-

tor, obtained through a nonperturbative solution of the equation, has the same infrared asymptotics as the one-loop expression: $\Pi(p) \approx C\alpha/p$, $C \approx 1 + 1/14N$. Thus, we have showed that a nontrivial IR fixed point persists in the nonperturbative solution. Moreover, quantitatively there is only slight difference between the one-loop result and the nonperturbative solution even at the number of fermions $N = 2$.

We then proceeded with studying the GNJL model in the symmetric phase with massless fermions. We solved an equation for the Yukawa vertex in the approximation where the full Bethe-Salpeter kernel is replaced by the planar one photon exchange graph with bare fermion-photon vertices. The obtained Yukawa vertex was used for calculation of the scalar composites propagator. The phase transition curve was determined from the condition of instability of the symmetric solution. We established the existence of light excitations (resonances) in the symmetric phase for values of $N > N_{cr} \approx 4$ ($\lambda < \lambda_c$), provided the four-fermion coupling ($g > 1/4$) is near its critical value along the critical curve (3). As $g < 1/4$ and N approaches N_{cr} from above the light excitations are absent and the situation resembles pure QED₃.

The field theoretical models, like QED₃ and the GNJL model, often appear in the long-wavelength limit of microscopic lattice models used for description of high- T_c samples. For instance, in a spin-charge separation ansatz for the t - J model, where spin is described by fermionic spinons and charge is described by bosonic holons (or vice versa), a ‘‘statistical’’ U(1) gauge interaction appears naturally in the theory along with four-fermion interactions (see, for example, Refs. [11], [20] and [25]). It was argued in Ref. [25] that QED₃, with fermions treated as spinons, might serve as a possible candidate for describing the undoped and underdoped cuprates. For physical $N (= 2)$ the chiral symmetry broken phase of QED₃ (with a dynamical mass generation) should correspond to an antiferromagnetic ordering in undoped cuprates [38], while the symmetric phase (for larger N) would describe some kind of a spin liquid.

Recently spin excitations (particle-hole bound states) have been observed in the normal state (and in the superconducting state) of underdoped and optimally doped cuprates such as YBa₂Cu₃O_{6+x} and La_{2-x}Sr_xCuO₄, where x is the amount of doping, see Ref. [39] and references therein. The dynamic susceptibility χ'' describing antiferromagnetic correlations near wave vector $Q = (\pi, \pi)$ has a broad peak whose energy comes down as the doping is reduced. The height of the peak increases as the doping is reduced and the antiferromagnetic transition approached.

As was proposed in Ref. [25], QED₃ could describe these particle-hole excitations. However, from the point of view of the present paper, pure QED₃ cannot be applied for describing such spin excitations because of absence of light resonances in the symmetric phase of the model. In our opinion, the GNJL model serves this purpose better since light excitations appear near the critical curve (3) on both sides. Moreover, the mass of resonances decreases as the phase transition is approached (along the trajectory N , or λ , is fixed and $g \uparrow g_c$) while their peaks become sharper as g approaches g_c . All these features are in qualitative agreement with the experimental picture if we assume that the four-fermion cou-

⁷A phase transition is by definition described by a nonanalytic behavior in the theory parameters (coupling constants, temperature, etc.). Therefore by analytical continuation one cannot go from one phase into another.

pling g depends on the doping in such a way that g increases when the doping is reduced.

A problem, however, is that, in case of cuprate superconductors, the physically relevant number of flavors equals two ($N=2$) which is less than $N_{\text{cr}} \sim 4$. This means that one would get the dynamically broken symmetry phase corresponding to the Néel ordered state at any doping in both QED₃ and the GNJL model. Kim and Lee [25] proposed a mechanism to lower N_{cr} (and make $N_{\text{cr}} < 2$) in pure QED₃ by taking into account the effect of doping which screens the time-component of the gauge field and halves N_{cr} , due to additional coupling of the gauge field to charged scalar fields. However, another way out of such a dilemma appears if we invoke the arguments of Appelquist *et al.* [40] that ladder SD equations usually overestimate the critical value N_{cr} . These authors suggest that the true critical value is $N_{\text{cr}} = 3/2$. Thus for the physical case of $N=2$ the spontaneous symmetry breaking does not occur and the system will be in the symmetric phase when the doping exceeds some

critical value. It would be quite interesting to find out a truncated set of SD equations giving such a small critical N_{cr} .

ACKNOWLEDGMENTS

We would like to acknowledge V. A. Miransky for useful and stimulating discussions and for bringing a paper (Ref. [40]) to our attention. We thank V. de la Incera, V. A. Miransky, and M. Winnink for carefully reading the manuscript. V. P. G. is grateful to the members of the Department of Physics of the Nagoya University, especially K. Yamawaki, for their hospitality during his stay there. His research has been supported in part by Deutscher Akademischer Austauschdienst (DAAD) grant, by the National Science Foundation (USA) under Grant No. PHY-9722059, and by the Grant-in-Aid of Japan Society for the Promotion of Science (JSPS) No. 11695030. He wishes to acknowledge JSPS for financial support.

APPENDIX A: BOX DIAGRAM

In this appendix we compute the box function B of Eq. (21). We start by contracting the γ 's in Eq. (15) and evaluate the traces. The result is

$$B(p^2, k^2, p \cdot k) = i \int_{M(2\pi)^3} \frac{d^3 r}{M(2\pi)^3} [b_1(p, k, r) + b_2(p, k, r) + b_3(p, k, r)], \quad (\text{A1})$$

where

$$b_1(p, k, r) = [-16(k \cdot r)^2 - 4k \cdot r(4k \cdot p - p^2) + 4(k^2 - 4k \cdot p)p \cdot r - 24k \cdot r p \cdot r - 16(p \cdot r)^2 - 24k \cdot r r^2 + 4(k^2 - 3k \cdot p + p^2)r^2 - 24p \cdot r r^2 - 12r^4] \frac{1}{(r+p)^2(r+k)^2(r+k)^2 r^2}, \quad (\text{A2})$$

and

$$b_2(p, k, r) = [-4k \cdot r p^2 + 4(2k \cdot p + p^2)p \cdot r + 8(p \cdot r)^2 + 4k \cdot r r^2 + 4(2k \cdot p + p^2)r^2 + 12p \cdot r r^2 + 4r^4] \frac{1}{r^2(r+p)^4(r+k)^2}, \quad (\text{A3})$$

$$b_3(p, k, r) = \frac{8k \cdot r p \cdot r - 4k \cdot p r^2 + 4k \cdot r r^2 + 4p \cdot r r^2 + 4r^4}{r^4(r+p)^2(r+k)^2}. \quad (\text{A4})$$

The traces have been performed with the help of FEYNALC [41]. Subsequently, we cancel the r dependence in the numerators of Eqs. (A2)–(A4) without shifting the integration variable. In this way the box function B can be expressed as follows (in Euclidean formulation):

$$B(-p^2, -k^2, -p \cdot k) = -2 \int_{E(2\pi)^3} \frac{d^3 r}{E(2\pi)^3} \left[\frac{2}{r^2(r+k)^2} + \frac{1}{r^2(r+k+p)^2} - \frac{4}{(r+p)^2(r+k)^2} + \frac{1}{(r+p)^4} + \frac{2k^2}{r^2(r+p)^2(r+k)^2} - \frac{k^2}{(r+p)^4(r+k+p)^2} + \frac{4k^2}{(r+k+p)^2(r+p)^2(r+k)^2} - \frac{2k^4}{r^2(r+k+p)^2(r+p)^2(r+k)^2} - \frac{2k \cdot p}{r^2(r+p)^2(r+k)^2} + \frac{2k \cdot p}{r^2(r+p)^2(r+k+p)^2} - \frac{2k \cdot p}{r^2(r+k+p)^2(r+k)^2} + \frac{4k \cdot p}{(r+k+p)^2(r+p)^2(r+k)^2} \right]$$

$$\begin{aligned}
& -\frac{4k^2k \cdot p}{r^2(r+k+p)^2(r+p)^2(r+k)^2} + \frac{2k \cdot r}{r^4(r+k)^2} - \frac{4k \cdot r}{r^2(r+p)^2(r+k)^2} + \frac{p^2}{r^4(r+k)^2} - \frac{p^2}{r^2(r+p)^4} \\
& -\frac{p^2}{r^2(r+k+p)^2(r+k)^2} + \frac{4p^2}{(r+k+p)^2(r+p)^2(r+k)^2} + \frac{k^2p^2}{r^2(r+p)^4(r+k+p)^2} \\
& -\frac{5k^2p^2}{r^2(r+k+p)^2(r+p)^2(r+k)^2} - \frac{4k \cdot pp^2}{r^2(r+k+p)^2(r+p)^2(r+k)^2} - \frac{2k \cdot rp^2}{r^4(r+p)^2(r+k)^2} \\
& -\frac{p^4}{r^4(r+p)^2(r+k)^2} - \frac{2p^4}{r^2(r+k+p)^2(r+p)^2(r+k)^2} - \frac{4p \cdot r}{r^2(r+p)^2(r+k)^2} - \frac{2p \cdot r}{r^2(r+k+p)^2(r+k)^2} \\
& -\frac{2p^2p \cdot r}{r^4(r+p)^2(r+k)^2} - \frac{2r^2}{(r+k+p)^2(r+p)^2(r+k)^2} \Big]. \tag{A5}
\end{aligned}$$

For the explicit calculation of the integral, only a handful of integrals are involved. These integrals are

$$\int_E \frac{d^3r}{(2\pi)^3} \left[\frac{1}{r^4} - \frac{k^2}{r^4(k+r)^2} \right] = 0, \tag{A6}$$

$$\int_E \frac{d^3r}{(2\pi)^3} \left[\frac{k^2p^2}{r^4(k+r)^2(p+r)^2} - \frac{k^2}{r^4(k+r)^2} \right] = \frac{k \cdot p}{8kp|k-p|}, \tag{A7}$$

$$\int_E \frac{d^3r}{(2\pi)^3} \frac{1}{r^2(k+r)^2} = \frac{1}{8k}, \tag{A8}$$

$$\int_E \frac{d^3r}{(2\pi)^3} \frac{1}{r^2(r+k)^2(r+p)^2} = \frac{1}{8kp|k-p|}, \tag{A9}$$

$$\begin{aligned}
& \int_E \frac{d^3r}{(2\pi)^3} \frac{1}{r^2(r+k)^2(r+p)^2(r+k+p)^2} \\
& = \frac{1}{8k \cdot p} \left[\frac{1}{kp|k-p|} - \frac{1}{kp|k+p|} \right], \tag{A10}
\end{aligned}$$

where $|k-p| = \sqrt{(k-p)^2}$. In the computation of the last integral (A10), we first have rewritten the left-hand side as the sum of four so-called triangle diagrams [diagrams of the form of Eq. (A9)], the subsequent integration is then straightforward.

The final result reads

$$B(-p^2, -k^2, -p \cdot k)$$

$$\begin{aligned}
& = \frac{1}{k} + \frac{1}{p} - \frac{k \cdot p}{4kp|k+p|} + \frac{k \cdot p}{4kp|k-p|} \\
& - \frac{(2k^2+kp+2p^2)}{2kp|k+p|} - \frac{(2k^2+kp+2p^2)}{2kp|k-p|} \\
& - \frac{(2k^4+5k^2p^2+2p^4)}{4(k \cdot p)kp|k+p|} + \frac{(2k^4+5k^2p^2+2p^4)}{4(k \cdot p)kp|k-p|}. \tag{A11}
\end{aligned}$$

If an additional integration over the angle between p and k follows we can simplify this equation because of the symmetry $p \cdot k \rightarrow -p \cdot k$. However, if we make use of this symmetry, we should take the principal value for angular integration, since the $1/p \cdot k$ singularity no longer explicitly cancels.

At various places, we need the following angular integrals:⁸

$$\int \frac{d\Omega}{4\pi} \frac{1}{|k-p|} = \frac{1}{\max(k,p)}, \tag{A12}$$

$$\int \frac{d\Omega}{4\pi} \frac{k \cdot p}{|k-p|} = \frac{k^2p^2}{3 \max(k^3, p^3)}, \tag{A13}$$

$$\int \frac{d\Omega}{4\pi} \frac{1}{(k+p)^2} = \frac{1}{2kp} \ln \frac{k+p}{|k-p|}, \tag{A14}$$

$$\int \frac{d\Omega}{4\pi} \frac{k \cdot p}{(k+p)^2} = \frac{1}{2} - \frac{(k^2+p^2)}{4kp} \ln \frac{k+p}{|k-p|}, \tag{A15}$$

⁸The angular measure is $\int d\Omega = 2\pi \int_0^\pi d\theta \sin \theta$, with $\cos \theta = k \cdot p / kp$.

$$\int \frac{d\Omega}{4\pi} \frac{(k \cdot p)^2}{(k+p)^2} = -\frac{(k^2+p^2)}{4} \times \left[1 - \frac{(k^2+p^2)}{2kp} \ln \frac{k+p}{|k-p|} \right], \quad (\text{A15})$$

and the Cauchy principal value integral

$$\mathcal{P} \int \frac{d\Omega}{4\pi} \frac{1}{k \cdot p |k-p|} = \frac{1}{kp \sqrt{k^2+p^2}} \ln \left(\frac{k+p+\sqrt{k^2+p^2}}{|k-p|+\sqrt{k^2+p^2}} \right). \quad (\text{A16})$$

Note that we use the notation $k = \sqrt{k^2}$, $p = \sqrt{p^2}$ for the scalar quantities in the right-hand side expressions of Eqs. (A12)–(A16).

Here, we compute also a general form of the integral over the kernel $K(p, k)$ given by Eq. (25), i.e.,

$$\begin{aligned} I(\delta) &= \int_0^\Lambda dk \frac{k^\delta}{p^\delta} K(p, k) \\ &= \int_0^1 dt t^\delta \left[-\frac{11t}{6} - \frac{1}{t} + \frac{2t^4+5t^2+2}{2t^2\sqrt{1+t^2}} \sinh^{-1} t \right] \\ &\quad + \int_{p/\Lambda}^1 dt t^{-1-\delta} \left[-\frac{11t}{6} - \frac{1}{t} + \frac{2t^4+5t^2+2}{2t^2\sqrt{1+t^2}} \sinh^{-1} t \right] \\ &= \int_0^1 dt (t^\delta + t^{-1-\delta}) \left[-\frac{11t}{6} - \frac{1}{t} + \frac{2t^4+5t^2+2}{2t^2\sqrt{1+t^2}} \sinh^{-1} t \right] \\ &\quad + \frac{2}{15} \frac{p}{\Lambda} \lim_{t \rightarrow 0} t^{2-\delta} + \mathcal{O}\left(\frac{p^2}{\Lambda^2}\right), \end{aligned} \quad (\text{A17})$$

with $-3 \leq \delta \leq 2$. Exact solutions exist when δ is an integer. In that case, one can make the transformation $t \rightarrow \sinh \ln s = (s^2-1)/2s$, after which the integral can be written as a sum of Spence functions. The result is

$$I(0) = \frac{\pi^2}{4} - \frac{5}{2}, \quad I(1) = \frac{\pi^2}{8} - \frac{23}{18}, \quad I(2) = \frac{\pi^2}{64} - \frac{1}{4}. \quad (\text{A18})$$

APPENDIX B: DERIVATION OF THE NONLOCAL GAUGE IN THE GNJL MODEL

In this appendix we derive the nonlocal gauge ξ in the GNJL model in order to set $A=1$, so that the WTI is satisfied if one uses the bare vertex approximation. For generality we consider the case of arbitrary dimensions D and in presence of the mass function B .

We introduce the nonlocal gauge function $\xi(k^2)$ by writing the full photon propagator in the form

$$e^2 D_{\mu\nu}(k) = - \left(g_{\mu\nu} - \zeta(k^2) \frac{k_\mu k_\nu}{k^2} \right) \frac{d(k^2)}{k^2}, \quad (\text{B1})$$

where $d(k^2) = e^2/[1 + \Pi(k^2)]$, $\zeta(k^2) = 1 - \xi(k^2)$. The SD equation for the fermion wave function renormalization A is given by

$$\begin{aligned} A(p^2) &= 1 + \frac{1}{p^2} \int \frac{d^D q}{(2\pi)^D} \frac{A(q^2)}{q^2 A^2(q^2) + B^2(q^2)} \\ &\quad \times \left\{ \frac{d(k^2)}{k^2} \left[(D-2)p \cdot q \right. \right. \\ &\quad \left. \left. + \left(p \cdot q - \frac{2(p^2 q^2 - (p \cdot q)^2)}{k^2} \right) \zeta(k^2) \right] \right. \\ &\quad \left. - p \cdot q [\Delta_S(k^2) + \Delta_P(k^2)] \right\}, \quad k = p - q. \end{aligned} \quad (\text{B2})$$

Introducing the variables $k^2 = x + y - 2\sqrt{xy} \cos \theta$, $x = p^2$, $y = q^2$, and performing the integration over all angles except the angle θ , we get

$$\begin{aligned} p^2[A(p^2) - 1] &= C_D \int_0^{\Lambda^2} dy \frac{y^{(D-2)/2} A(y)}{y A^2(y) + B^2(y)} \int_0^\pi d\theta \sin^{D-2} \theta \\ &\quad \times \left\{ d(k^2) \left[\frac{\sqrt{xy} \cos \theta [D-2 + \zeta(k^2)]}{k^2} \right. \right. \\ &\quad \left. \left. - 2xy \frac{\sin^2 \theta}{k^4} \zeta(k^2) \right] - \sqrt{xy} \cos \theta [\Delta_S(k^2) \right. \\ &\quad \left. + \Delta_P(k^2)] \right\}, \end{aligned} \quad (\text{B3})$$

where $C_D^{-1} = 2^D \pi^{(D+1)/2} \Gamma[(D-1)/2]$. Following the works by Kugo *et al.* and Kondo *et al.* [42] (see also Ref. [43]) we perform now the θ integration by parts in terms containing the first power of $\cos \theta$:

$$\begin{aligned} p^2(A(p^2) - 1) &= -\frac{C_D}{D-1} \int_0^{\Lambda^2} dy \frac{y^{(D-2)/2} A(y) (2xy)}{y A^2(y) + B^2(y)} \\ &\quad \times \int_0^\pi d\theta \sin^{D-2} \theta \left\{ \frac{1}{z^{D-1}} [[z^{D-2} d(z) \zeta(z)]' \right. \\ &\quad \left. - (D-2) z^{D-3} [d(z) - z d'(z)] \right. \\ &\quad \left. - [\Delta_S(z) + \Delta_P(z)]' \right\}, \end{aligned} \quad (\text{B4})$$

where the prime denotes the differentiation with respect to $z = k^2$.

The requirement $A(p^2) = 1$ is fulfilled by choosing $\zeta(z)$ such that the expression in curly brackets in Eq. (B4) vanishes. This gives the first order differential equation for $\zeta(z)$ which is easy to integrate,

$$\begin{aligned} \zeta(z) &= \frac{D-2}{z^{D-2}d(z)} \int_0^z dt t^{D-3} [d(t) - td'(t)] \\ &+ \frac{1}{z^{D-2}d(z)} \int_0^z dt t^{D-1} [\Delta_S(z) + \Delta_P(z)]' \end{aligned} \quad (\text{B5})$$

(the integration constant was fixed by requiring $[z^{D-2}d(z)\zeta(z)]|_{z=0}=0$ in order to eliminate the singularity at $z=0$ in $\zeta(z)$). The last equation finally leads to the following expression for $\xi(z)$:

$$\begin{aligned} \xi(z) &= D-1 - \frac{(D-1)(D-2)}{z^{D-2}d(z)} \int_0^z dt t^{D-3} d(t) \\ &- \frac{1}{z^{D-2}d(z)} \int_0^z dt t^{D-1} [\Delta_S(z) + \Delta_P(z)]'. \end{aligned} \quad (\text{B6})$$

For $D=3$ we take

$$d(k) = \frac{e^2}{1 + \Pi(k)} \simeq \frac{8}{NC} k, \quad k \ll \alpha, \quad (\text{B7})$$

with $k = \sqrt{k^2}$, and assume the following form for scalar propagators in the symmetric phase and near the critical line [see Eq. (76)]:

$$\Delta_S(k) = \Delta_P(k) = -\frac{a}{\Lambda} \left(\frac{\Lambda^2}{k^2} \right)^\gamma, \quad (\text{B8})$$

where a is some constant and the power $0 < \gamma < 1$ [Eq. (B8) is verified *a posteriori* when solving the SD Eq. (40) for the scalar propagator]. We obtain, from Eq. (B6),

$$\xi(k) = \frac{2}{3} - \frac{NC\gamma a}{4(2-\gamma)} \left(\frac{\Lambda}{k} \right)^{2\gamma-1}. \quad (\text{B9})$$

In absence of the four-fermion interaction we get the famous nonlocal gauge $\xi=2/3$ [16]. Carena *et al.* [21] have included exchanges by the bare scalar propagators what corresponds to taking $\gamma=1/2$, $a=4/N$ ($C=1$ in their leading order of the $1/N$ approximation for the photon vacuum polarization). Equation (B9) then gives $\xi=1/3$ in accordance with their findings.

Our Eq. (74) for the scalar propagator gives the exponent $\gamma=\omega/2$ and contribution due to the exchange of scalars into $\xi(k)$ becomes suppressed (since $\omega < 1$) and we are left with Nash's nonlocal gauge $\xi=2/3$.

APPENDIX C: APPROXIMATION FOR THE YUKAWA VERTEX

As was mentioned in Sec. III, in order to resolve the angular dependence of the Yukawa vertex function F_1 , we expand it, together with the kernels of Eqs. (48) and (49), in Legendre polynomials P_n . We write

$$F_1(p+q, p) = \sum_{n=0}^{\infty} f_n(p, q) P_n(\cos \alpha),$$

$$\frac{1}{|k-p|} = \sum_{n=0}^{\infty} N_n(k, p) P_n(\cos \beta), \quad (\text{C1})$$

$$\frac{k^2 + q \cdot k}{(k+q)^2} = \sum_{n=0}^{\infty} a_n(k, q) P_n(\cos \gamma), \quad (\text{C2})$$

where $\cos \alpha = p \cdot q / pq$, $\cos \beta = p \cdot k / pk$, and $\cos \gamma = q \cdot k / qk$. The Legendre polynomials P_n satisfy

$$\begin{aligned} \int \frac{d\Omega}{4\pi} P_m(\cos \alpha) P_n(\cos \alpha) &= \frac{1}{2} \int_{-1}^1 dx P_m(x) P_n(x) \\ &= \frac{\delta_{mn}}{2n+1}. \end{aligned} \quad (\text{C3})$$

With the above defined expansions, and by making use of the identity

$$\begin{aligned} \frac{1}{4\pi} \int_0^\pi d\alpha \sin \alpha \int_0^{2\pi} d\theta P_n(\cos \alpha) P_l(\cos \beta) \\ = \frac{\delta_{nl}}{2l+1} P_l(\cos \gamma), \end{aligned} \quad (\text{C4})$$

where $\cos \beta = \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \theta$, Eq. (48) for the Yukawa vertex can be represented as the set of equations for harmonics f_l :

$$\begin{aligned} f_l(p, q) &= \delta_{0l} + \lambda \int_0^\Lambda dk N_l(k, p) \\ &\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{lmn} a_m(k, q) f_n(k, q), \end{aligned} \quad (\text{C5})$$

where

$$\begin{aligned} C_{lmn} &\equiv \frac{1}{2} \int_{-1}^1 dx P_l(x) P_m(x) P_n(x) \\ &= \frac{\left(\frac{1}{2}\right)_{s-l} \left(\frac{1}{2}\right)_{s-m} \left(\frac{1}{2}\right)_{s-n} s!}{(s-l)!(s-m)!(s-n)! \left(\frac{1}{2}\right)_s (2s+1)}, \end{aligned} \quad (\text{C6})$$

where $2s = l + m + n$ and $(a)_k \equiv \Gamma(a+k)/\Gamma(a)$. The coefficients C_{lmn} are zero unless $l + m + n = 2s$ is even and a triangle with sides l, m, n exists, i.e., $|l-m| \leq n \leq l+m$.⁹ Furthermore, Eq. (49) can be written as

$$\Pi_S(q) = \frac{2N}{\pi^2} \int_0^\Lambda dk \sum_{n=0}^{\infty} \frac{a_n(k, q) f_n(k, q)}{2n+1}. \quad (\text{C7})$$

⁹We thank L. P. Kok for pointing out the paper by Askey *et al.* [31].

Within the approximation (51), i.e., keeping the zero-order term f_0 only in right-hand sides of Eqs. (C5) and (C7), we get

$$f_0(p, q) = 1 + \lambda \int_0^\Lambda dk N_0(k, p) a_0(k, q) f_0(k, q), \quad (\text{C8})$$

$$\Pi_S(q) = \frac{2N}{\pi^2} \int_0^\Lambda dk a_0(k, q) f_0(k, q). \quad (\text{C9})$$

The functions N_0 and a_0 are straightforwardly obtained from inverting Eqs. (C1) and (C2). This gives

$$N_0(k, p) = \frac{\theta(k-p)}{k} + \frac{\theta(p-k)}{p}, \quad (\text{C10})$$

and

$$\begin{aligned} a_0(k, q) &= \int \frac{d\Omega}{4\pi} \frac{(k^2 + q \cdot k)}{(k+q)^2} \\ &= a_{\text{IR}}(k, q) \theta(q-k) + a_{\text{UV}}(k, q) \theta(k-q), \end{aligned} \quad (\text{C11})$$

$$a_{\text{IR}}(k, q) \equiv \frac{1}{2} + \frac{(k^2 - q^2)}{4qk} \ln \frac{k+q}{q-k},$$

$$a_{\text{UV}}(k, q) \equiv \frac{1}{2} + \frac{(k^2 - q^2)}{4qk} \ln \frac{k+q}{k-q}. \quad (\text{C12})$$

In order to be able to solve the equations for F_{IR} and F_{UV} given by Eq. (52), we approximate the functions a_{IR} and a_{UV} as follows:

$$\begin{aligned} a_{\text{IR}}(k, q) &\approx \frac{k^2}{2q^2}, \quad a_{\text{UV}}(k, q) \approx 1 - \frac{q^2}{2k^2}, \\ a_{\text{IR}}(q, q) &= a_{\text{UV}}(q, q) = \frac{1}{2}. \end{aligned} \quad (\text{C13})$$

The validity of this approximation is addressed in Sec. IV A.

The lowest order harmonic f_0 of the particular Legendre expansion given in Eq. (50) is expressed in terms of the

so-called infrared (IR) function F_{IR} and the ultraviolet (UV) function F_{UV} , see Eq. (52). These functions describe the following asymptotic behavior of the Yukawa vertex:

$$\lim_{p \gg q} \Gamma_S(p+q, p) = \mathbf{1} \lim_{p \gg q} F_{\text{UV}}(p, q), \quad (\text{C14})$$

$$\lim_{q \gg p} \Gamma_S(p+q, p) = \mathbf{1} \lim_{q \gg p} F_{\text{IR}}(p, q). \quad (\text{C15})$$

The fact that both these asymptotic limits of Γ_S are described by f_0 through F_{IR} and F_{UV} guarantees the validity of the approximation (51). This crucial point is explained in more detail in Ref. [24], where this approximation is referred to as ‘‘the two-channel approximation.’’

Then, by making use of Eqs. (52), (C10), and (C13) we get

$$\begin{aligned} (p < q) F_{\text{IR}}(p, q) &= 1 + \lambda \int_0^p dk \frac{k^2}{2pq^2} F_{\text{IR}}(k, q) \\ &+ \lambda \int_p^q dk \frac{k}{2q^2} F_{\text{IR}}(k, q) + \lambda \int_q^\Lambda dk \frac{1}{k} \\ &\times \left(1 - \frac{q^2}{2k^2} \right) F_{\text{UV}}(k, q), \end{aligned} \quad (\text{C16})$$

$$\begin{aligned} (p > q) F_{\text{UV}}(p, q) &= 1 + \lambda \int_0^q dk \frac{k^2}{2pq^2} F_{\text{IR}}(k, q) \\ &+ \lambda \int_q^p dk \frac{1}{p} \left(1 - \frac{q^2}{2k^2} \right) F_{\text{UV}}(k, q) + \lambda \int_p^\Lambda dk \frac{1}{k} \\ &\times \left(1 - \frac{q^2}{2k^2} \right) F_{\text{UV}}(k, q), \end{aligned} \quad (\text{C17})$$

and for the scalar vacuum polarization (C9) we can derive Eq. (53). The integral Eqs. (C16) and (C17) are equivalent to the second order differential equations given in Eqs. (54) and (55), with the four boundary conditions (56) and (57).

-
- [1] T. Appelquist, M. Bowick, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. D **33**, 3704 (1986).
[2] T. Appelquist, D. Nash, and L. C. R. Wijewardhana, Phys. Rev. Lett. **60**, 2575 (1988).
[3] T. Maskawa and H. Nakajima, Prog. Theor. Phys. **52**, 1326 (1974); R. Fukuda and T. Kugo, Nucl. Phys. **B117**, 250 (1976).
[4] P. I. Fomin, V. P. Gusynin, and V. A. Miransky, Phys. Lett. **78B**, 136 (1978); P. I. Fomin, V. P. Gusynin, V. A. Miransky, and Yu. A. Sitenko, Riv. Nuovo Cimento **6**, N5, 1 (1983).
[5] V. A. Miransky and K. Yamawaki, Phys. Rev. D **55**, 5051 (1997).

- [6] V. P. Gusynin, V. A. Miransky, and A. V. Shpagin, Phys. Rev. D **58**, 085023 (1998).
[7] T. Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982).
[8] T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. Lett. **77**, 1214 (1996).
[9] F. R. Brown, H. Chen, N. H. Christ, Z. Dong, R. D. Mawhinney, W. Shafer, and A. Vaccarina, Phys. Rev. D **46**, 5655 (1992).
[10] Y. Iwasaki, K. Kanaya, S. Sakai, and T. Yoshié, Phys. Rev. Lett. **69**, 21 (1992).
[11] N. Dorey and N. E. Mavromatos, Phys. Lett. B **250**, 107

- (1990); A. Kovner and B. Rosenstein, Phys. Rev. B **42**, 4748 (1990).
- [12] I. J. R. Aitchison and N. E. Mavromatos, Phys. Rev. B **53**, 9321 (1996); K. Farakos and N. E. Mavromatos, Int. J. Mod. Phys. B **12**, 809 (1998).
- [13] M. R. Pennington and D. Walsh, Phys. Lett. B **253**, 246 (1991); D. C. Curtis, M. R. Pennington, and D. Walsh, *ibid.* **295**, 313 (1992).
- [14] R. D. Pisarski, Phys. Rev. D **44**, 1866 (1991).
- [15] P. Maris, Phys. Rev. D **54**, 4049 (1996).
- [16] D. Nash, Phys. Rev. Lett. **62**, 3024 (1989).
- [17] I. J. Aitchison, N. E. Mavromatos, and D. McNeill, Phys. Lett. B **402**, 154 (1997); K. Kondo and T. Murakami, *ibid.* **410**, 257 (1997).
- [18] V. P. Gusynin, A. H. Hams, and M. Reenders, Phys. Rev. D **53**, 2227 (1996).
- [19] N. E. Mavromatos and J. Papavassiliou, Phys. Rev. D **60**, 125008 (1999).
- [20] G. W. Semenoff and L. C. R. Wijewardhana, Phys. Rev. Lett. **63**, 2633 (1989); N. Dorey and N. E. Mavromatos, Nucl. Phys. **B386**, 614 (1992); K. Farakos and N. E. Mavromatos, Phys. Rev. B **57**, 3017 (1998).
- [21] M. Carena, T. E. Clark, and C. E. M. Wagner, Nucl. Phys. **B356**, 117 (1991).
- [22] B. Rosenstein, B. J. Warr, and S. H. Park, Phys. Rev. Lett. **62**, 1433 (1989).
- [23] T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. D **44**, 871 (1991).
- [24] V. P. Gusynin and M. Reenders, Phys. Rev. D **57**, 6356 (1998).
- [25] D. H. Kim and P. A. Lee, Ann. Phys. (N.Y.) **272**, 130 (1999).
- [26] K. Farakos, G. Koutsoumbas, and N. E. Mavromatos, Int. J. Mod. Phys. B **12**, 2475 (1998).
- [27] R. Karplus and M. Neuman, Phys. Rev. **80**, 380 (1950).
- [28] C. J. Burden, J. Praschifka, and G. D. Roberts, Phys. Rev. D **46**, 2695 (1992).
- [29] M. Hashimoto, Prog. Theor. Phys. **100**, 781 (1998).
- [30] C. J. Burden, Nucl. Phys. **B387**, 419 (1992).
- [31] R. Askey, T. H. Koornwinder, and M. Rahman, J. London Math. Soc. (2) **33**, 133 (1986).
- [32] S. Hands, A. Kocić, and J. B. Kogut, Phys. Lett. B **273**, 111 (1991).
- [33] Y. Kikukawa and K. Yamawaki, Phys. Lett. B **234**, 497 (1990).
- [34] D. J. Thouless, Ann. Phys. (N.Y.) **10**, 553 (1960).
- [35] S. Hands, A. Kocić, and J. B. Kogut, Ann. Phys. (N.Y.) **224**, 29 (1993).
- [36] M. Carena, T. E. Clark, and C. E. M. Wagner, Phys. Lett. B **259**, 128 (1991).
- [37] T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. Lett. **75**, 2081 (1995).
- [38] J. B. Marston, Phys. Rev. Lett. **64**, 1166 (1990).
- [39] H. F. Fong, P. Bourges, Y. Sidis, L. P. Regnault, J. Bossy, A. Ivanov, D. L. Milius, I. A. Aksay, and B. Keimer, Phys. Rev. B **61**, 14773 (2000).
- [40] T. Appelquist, A. G. Cohen, and M. Schmaltz, Phys. Rev. D **60**, 045003 (1999).
- [41] R. Mertig, M. Bohm, and A. Denner, Comput. Phys. Commun. **64**, 345 (1991).
- [42] T. Kugo and M. G. Mitchard, Phys. Lett. B **282**, 162 (1992); K.-I. Kondo, T. Ebihara, T. Iizuka, and E. Tanaka, Nucl. Phys. **B434**, 85 (1995).
- [43] E. H. Simmons, Phys. Rev. D **42**, 2933 (1990).