Boole and Bell inequality

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Abstract. We discuss the relation between Bell's and Boole's inequality. We apply both to the analysis of measurement results in idealized Einstein-Podolsky-Rosen-Bohm experiments. We present a local realist model that violates Bell's and Boole's inequality due to the absence of Boole's one-to-one correspondence between the two-valued variables of the mathematical description and the two-valued measurement results.

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BELL INEQUALITY

We consider a model of Einstein-Podolsky-Rosen-Bohm (EPRB) experiments [1] used by Bell when presenting his inequality [2]. In this model a source produces pairs of spin-1/2 particles, prepared in the singlet state $|\Psi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/2^{1/2}$. The two particles with opposite spins move in free space and in opposite directions. Components of the spins σ_1 and σ_2 of the individual particles are measured by means of Stern-Gerlach magnets. Representing the directions of the two magnets by the unit vectors **a** and **b**, respectively, quantum theory (QT) yields $E_1(\mathbf{a}) = \langle\Psi|\sigma_1\cdot\mathbf{a}|\Psi\rangle = 0$, $E_2(\mathbf{b}) = \langle\Psi|\sigma_2\cdot\mathbf{b}|\Psi\rangle = 0$ and $E(\mathbf{a},\mathbf{b}) = \langle\Psi|\sigma_1\cdot\mathbf{a}\sigma_2\cdot\mathbf{b}|\Psi\rangle = -\mathbf{a}\cdot\mathbf{b}$, where $\sigma_i = (\sigma_i^x,\sigma_i^y,\sigma_i^z)$ are the three Pauli spin operators of particle $i=1,2$. QT yields statistical estimates for E_1, E_2 and E_{12} and cannot say anything about individual measurements [3]. Nevertheless, QT predicts that, if measurement of the component $\sigma_1\cdot\mathbf{a}$ yields the value + 1, then measurement of $\sigma_2\cdot\mathbf{a}$ must yield the value -1 and vice versa. The fundamental question is how to relate the statistical results of QT and the individual measurements.

Bell made the following assumptions in constructing his model and deriving his inequality [2]:

1. $A(\mathbf{a},\lambda) = \pm 1$ and $B(\mathbf{b},\lambda) = \pm 1$, where A (B) denotes the result of measuring $\sigma_1\cdot\mathbf{a}$ ($\sigma_2\cdot\mathbf{b}$) and λ denotes a variable or a set of variables which only depend on the preparation (source) and not on the measurement (magnet settings) of the spin components. Note that this assumption already includes the hypothesis that the orientation of one magnet does not influence the measurement result obtained with the other magnet (often referred to as the locality condition).
2. If $\rho(\lambda)$ is the probability distribution of λ ($\int\rho(\lambda)d\lambda = 1$) then the expectation

value of the product of the two components $\sigma_1 \cdot \mathbf{a}$ and $\sigma_2 \cdot \mathbf{b}$ can be written as $P(\mathbf{a},\mathbf{b}) = \int d\lambda \rho(\lambda) A(\mathbf{a},\lambda) B(\mathbf{b},\lambda)$. Note that one could also assume variables λ' and λ'' depending on the characteristics of the instruments on both sides. Averaging over these instrument dependent variables would result in new variables having values between -1 and +1. However, this is only the case if λ' and λ'' are completely independent. For example, if λ' and λ'' are sets of variables including the detection times, used for coincidence measurements in a laboratory experiment, then assumption 2 does not hold [4].

3. $A(\mathbf{a},\lambda) = -B(\mathbf{a},\lambda)$ so that $P(\mathbf{a},\mathbf{b}) = -\int d\lambda \rho(\lambda) A(\mathbf{a},\lambda) A(\mathbf{b},\lambda)$. This assumption follows from the observation that $P(\mathbf{a},\mathbf{b}) = \int d\lambda \rho(\lambda) A(\mathbf{a},\lambda) B(\mathbf{b},\lambda)$ reaches -1 at $\mathbf{a} = \mathbf{b}$ only if $A(\mathbf{a},\lambda) = -B(\mathbf{a},\lambda)$. Note that $P(\mathbf{a},\mathbf{a}) = -1$ if and only if $A(\mathbf{a},\lambda) = -B(\mathbf{a},\lambda)$, making both these assumptions equivalent. Hence, what Bell assumed is that the results of the measurements at both sides of the source can be represented by one and the same symbol "A" that depends only on the respective magnet setting and on λ. Moreover, also the measurement outcomes of an experiment with another setting of (only one of) the magnets, can be represented by the same symbol "A".

Using the above hypotheses and considering a third unit vector $\mathbf{c}$ Bell derived the inequality [2]

$$|P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq 1 + P(\mathbf{b},\mathbf{c}), \tag{1}$$

which is violated for certain magnet settings $\mathbf{a},\mathbf{b},\mathbf{c}$ if $P(\mathbf{a},\mathbf{b})$ is replaced by $E(\mathbf{a},\mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$, the quantum theoretical two-particle expectation value describing the averaged two-particle correlations obtained in EPRB laboratory experiments. Note that 1, 2 and 3 are sufficient conditions for the Bell inequality to be obeyed. Hence, if the Bell inequality is obeyed then one cannot say anything about the validity of the assumptions, but if it is violated then one can say that at least one of the assumptions must be false, thereby refuting Bell's model.

BOOLE INEQUALITY

We consider two-valued variables $S(x,n) = \pm 1$ where x can be considered to represent the orientations $\mathbf{a},\mathbf{b},\mathbf{c}$ of the magnets in an EPRB experiment and $n = 1,\ldots N$ simply numbers the measurements in an experimental run. From the variables $S(x,n)$ with $x = \mathbf{a},\mathbf{b},\mathbf{c}$ we compute the averages $F_{\mathbf{a},\mathbf{b}} = \sum_{n=1}^{N} S(\mathbf{a},n)S(\mathbf{b},n)/N$, $F_{\mathbf{a},\mathbf{c}} = \sum_{n=1}^{N} S(\mathbf{a},n)S(\mathbf{c},n)/N$ and $F_{\mathbf{b},\mathbf{c}} = \sum_{n=1}^{N} S(\mathbf{b},n)S(\mathbf{c},n)/N$. According to Boole it is impossible to violate

$$|F_{\mathbf{a},\mathbf{b}} \pm F_{\mathbf{a},\mathbf{c}}| \leq 1 \pm F_{\mathbf{b},\mathbf{c}}, \tag{2}$$

if there is a one-to-one correspondence between the two-valued variables $S(\mathbf{a},n)$, $S(\mathbf{b},n)$, $S(\mathbf{c},n)$ of the mathematical description and each triple $\{X(\mathbf{a},n), X(\mathbf{b},n), X(\mathbf{c},n)\}$ of binary data collected in the experimental run denoted by n. This one-to-one correspondence is a necessary and sufficient condition for the inequality to be obeyed. Note that inequalities Eq. (1) and Eq. (2) have the same structure.

BELL INEQUALITY TESTS

In a typical ideal EPRB experiment three runs are performed in which N detection events are collected on both sides (referred to by 1 and 2) of the source. The outcomes of the detection events take the values + 1 or -1 and are represented by the symbol X. This results in the three data sets $\Gamma_{\mathbf{a},\mathbf{b}}(n) = \{X(\mathbf{a},n,1), X(\mathbf{b},n,2)|n = 1,\ldots,N\}$, $\widetilde{\Gamma}_{\mathbf{a},\mathbf{c}}(\widetilde{n}) = \{\widetilde{X}(\mathbf{a},\widetilde{n},1), \widetilde{X}(\mathbf{c},\widetilde{n},2)|\widetilde{n} = 1,\ldots,N\}$ and $\widehat{\Gamma}_{\mathbf{b},\mathbf{c}}(\widehat{n}) = \{\widehat{X}(\mathbf{b},\widehat{n},1), \widehat{X}(\mathbf{c},\widehat{n},2)|\widehat{n} = 1,\ldots,N\}$. Note that in real experiments the measurement outcomes are also labeled by the time of measurement but for simplicity we omit this label here. Using these data sets for testing the validity of Bell's inequality Eq. (1) and of the structurally equivalent Boole inequality Eq. (2), requires making the following assumptions:

1. The same symbol X can be used for all the data collected in the three runs. This results in the data set $\Upsilon(n,\widetilde{n},\widehat{n}) = \{X(\mathbf{a},n,1), X(\mathbf{a},\widetilde{n},1), X(\mathbf{b},n,2), X(\mathbf{b},\widetilde{n},1), X(\mathbf{c},\widetilde{n},2), X(\mathbf{c},\widehat{n},2)|n,\widetilde{n},\widehat{n} = 1,\ldots,N\}$.
2. The data can be rearranged such that $X(\mathbf{a},n,1) = X(\mathbf{a},\widetilde{n},1)$, $X(\mathbf{b},\widehat{n},1) = X(\mathbf{b},n,1)$ and $X(\mathbf{c},\widetilde{n},2) = X(\mathbf{c},\widehat{n},2) = X(\mathbf{c},n,2)$. This results in the data set $\Upsilon'(n) = \{X(\mathbf{a},n,1), X(\mathbf{b},n,2), X(\mathbf{b},n,1), X(\mathbf{c},n,2)|n = 1,\ldots,N\}$, a data set containing quadruples, not yet triples, as used in the derivation of Bell's inequality and as required by Boole for his inequality to be obeyed. Reduction to a set of triples requires the extra assumption:
3. $X(\mathbf{b},n,1) = X(\mathbf{b},n,2)$

Since the data in an EPRB experiment are not collected as one set of triples but as three sets of pairs, at least one of the assumptions 1, 2 or 3 is false in case a violation of Boole's inequality Eq. (2) is found. In other words, if the data sets collected in an EPRB experiment satisfy these three conditions, the one-to-one correspondence between the two-valued variables in the mathematical description and the observed two-valued experimental data is guaranteed, and hence Boole's and thus also Bell's inequality are satisfied. If the Bell inequality is violated then at least one of the sufficient conditions 1, 2 or 3 to derive the Bell inequality is false, but then also at least one of the assumptions listed above is false.

LOCAL REALIST MODEL VIOLATING BELL'S INEQUALITY

We consider testing an allergy to alcohol that strikes persons in different ways depending on circumstances such as place of birth, place of diagnosis etc. [5, 6]. We consider three groups with persons born in Austria (denoted by $\mathbf{o} = \mathbf{a}$), in Brazil ($\mathbf{o} = \mathbf{b}$) and in Canada ($\mathbf{o} = \mathbf{c}$), respectively. The allergy tests take place in three doctor's offices located in Lille (denoted by $l = 1$), Lyon ($l = 2$) and Paris ($l = 3$), respectively.

The allergy tests performed in the doctor's offices are identical and consist of serving the persons a glass of wine diluted with water from the tap. When a person is allergic he or she gets a red rash and gets assigned a value $A^l_{\mathbf{o}}(w,n) = +1$, otherwise $A^l_{\mathbf{o}}(w,n) = -1$, where w refers to the fact that a diluted glass of wine is served to make the allergy test and $n = 1,\ldots,N$ numbers the examinations. Eventually the doctors could use additional labels that they think to be relevant for the outcome of their observations.

Assume that on even days the tap water contains no additives in Lille, iron in Lyon and chlorine in Paris. On odd days the tap water contains fluorine and iron in Lille, chlorine and fluorine in Lyon and fluorine and iron in Paris. This information is not known to the doctors, hence they assume that they perform identical allergy tests. Also not known to the doctors is that persons born in Austria are allergic to alcohol, not allergic to chlorine or iron, and also not allergic if alcohol and fluorine are present at the same time. Persons born in Brazil are allergic to alcohol, not allergic to fluorine or chlorine, and also not allergic if alcohol and iron are both present. Persons born in Canada are allergic to fluorine only. The results of all possible examinations on even days are: $A^1_{\mathbf{a}}(w,n) = A^2_{\mathbf{a}}(w,n) = A^3_{\mathbf{a}}(w,n) = +1$, $A^1_{\mathbf{b}}(w,n) = A^3_{\mathbf{b}}(w,n) = +1$, $A^2_{\mathbf{o}}(w,n) = -1$, and $A^1_{\mathbf{c}}(w,n) = A^2_{\mathbf{c}}(w,n) = A^3_{\mathbf{c}}(w,n) = -1$. On odd days they are simply the opposite. Note that the believe of the doctors that they perform identical allergy tests leads to the hypothesis $A^1_{\mathbf{o}}(w,n) = A^2_{\mathbf{o}}(w,n) = A^3_{\mathbf{o}}(w,n)$ and thus $A^l_{\mathbf{o}}(w,n)A^{l'}_{\mathbf{o}}(w,n) = +1$, which is equivalent to the assumption $P(\mathbf{a},\mathbf{a}) = -1$ made by Bell in case of the EPRB experiment.

In the first variation of the experiment the doctor in Lille examines only persons of type **a**, the doctor in Lyon only persons of type **b** and the doctor in Paris only persons of type **c**. The doctors perform the examinations on randomly chosen but identical dates and each doctor only sees one person per day. In other words, each day a triple of diagnoses is made. On any given day of examination numbered by n they write down their diagnosis and then, after many exams, concatenate the results and form the sum of pair-products of exam outcomes $\Gamma(w,n) = A^1_{\mathbf{a}}(w,n)A^2_{\mathbf{b}}(w,n) + A^1_{\mathbf{a}}(w,n)A^3_{\mathbf{c}}(w,n) + A^2_{\mathbf{b}}(w,n)A^3_{\mathbf{c}}(w,n)$. By inserting all possible values for the examination outcomes we find $\Gamma(w,n) \geq -1$. For the average over all examinations we have $\Gamma(w) = \langle\Gamma(w,n)\rangle = \sum_{n=1}^{N}\Gamma(w,n)/N \geq -1$. This non-trivial inequality gives conditions for the frequencies of the concurrence of certain values of $A^1_{\mathbf{a}}(w,n), A^2_{\mathbf{b}}(w,n)$ etc. Boole calls them “conditions of possible experience". Note that the experimental outcomes have been determined from an experimental procedure in a scientific way and are therefore always possible. What may not be possible is to establish the one-to-one correspondence of Boole’s two-valued variables to the two-valued experimental outcomes. In this example, we may indeed regard the various $A^l_{\mathbf{o}}(w,n) = \pm 1$ as the elements of Boole’s logic to which the actual experiments can be mapped. As shown by Boole, this is a sufficient condition for the inequality $\Gamma(w) \geq -1$ to be obeyed. We may in this case omit all the labels except for those designating the birth place and still obtain an inequality that never can be violated: $\langle A_{\mathbf{a}}(w)A_{\mathbf{b}}(w)\rangle + \langle A_{\mathbf{a}}(w)A_{\mathbf{c}}(w)\rangle + \langle A_{\mathbf{b}}(w)A_{\mathbf{c}}(w)\rangle \geq -1$. The reason is that the three products are deduced from sequences of measurement outcomes obtained in triples.

In the second variation of the experiment, the doctor in Lille examines all persons of type **a** and **b** and the one in Lyon all persons of type **b** and **c**. The doctors perform the examinations on randomly chosen but identical dates and each doctor sees only one person per day. In other words, three days are required to obtain the three pairs of diagnoses $(A^1_{\mathbf{a}}, A^2_{\mathbf{b}})$, $(A^1_{\mathbf{a}}, A^2_{\mathbf{c}})$ and $(A^1_{\mathbf{b}}, A^2_{\mathbf{c}})$. Since the days of examination are chosen randomly all these diagnoses can be labeled by the same number n. The doctors are convinced that neither the date of examination nor the location (Lille or Lyon) has any influence and therefore denote the persons only by their place of birth. They find

$\Gamma(w) = \langle A_{\mathbf{a}}(w)A_{\mathbf{b}}(w)\rangle + \langle A_{\mathbf{a}}(w)A_{\mathbf{c}}(w)\rangle + \langle A_{\mathbf{b}}(w)A_{\mathbf{c}}(w)\rangle = -3$ and notice that the single outcomes of $A_{\mathbf{a}}(w), A_{\mathbf{b}}(w)$ and $A_{\mathbf{c}}(w)$ are randomly equal to ± 1. This latter fact completely baffles them. How can the single outcomes be entirely random while the products are not random at all and how can a Boole inequality be violated? They conclude that there must be some influence at a distance going on and the outcomes depend on the exams in both Lille and Lyon such that a single outcome manifests itself randomly in one city and that the outcome in the other city is then always of opposite sign.

However, this observation can also be explained differently. Although not known to the doctors beforehand, the allergy is time- and city-dependent as described above. Obviously for measurements on random dates we have the outcome that $A_{\mathbf{a}}(w), A_{\mathbf{b}}(w)$ and $A_{\mathbf{c}}(w)$ are randomly equal to ± 1 while at the same time $\Gamma(w,n) = -3$ and therefore $\Gamma(w) = -3$. However, in order to deal with Boole's elements of logic, we need to add the city labels to obtain $\Gamma(w) = \langle A^1_{\mathbf{a}}(w)A^2_{\mathbf{b}}(w)\rangle + \langle A^1_{\mathbf{a}}(w)A^2_{\mathbf{c}}(w)\rangle + \langle A^1_{\mathbf{b}}(w)A^2_{\mathbf{c}}(w)\rangle \geq -3$ and the inequality is of the trivial kind because there is no one-to-one correspondence between the two-valued variables of the mathematical description and the two-valued diagnoses. The label n can be omitted because the signs of the diagnoses are reversed on even and odd days leaving the products unchanged. Including the city labels the doctors realize that $A^1_{\mathbf{b}}(w,n) = -A^2_{\mathbf{b}}(w,n)$, totally against their expectations. Contacting the water delivering company can however resolve this mystery. After contacting the water delivery company the two doctors realized that their diagnosis is city dependent because the water they used is different. Hence, they could also have written $\langle A_{\mathbf{a}}(w^1)A_{\mathbf{b}}(w^2)\rangle + \langle A_{\mathbf{a}}(w^1)A_{\mathbf{c}}(w^2)\rangle + \langle A_{\mathbf{b}}(w^1)A_{\mathbf{c}}(w^2)\rangle \geq -3$. Interpreting the variable w as playing the role of the variable λ in Bell's inequality, it is clear that the use of the same label λ for both measurement sides is necessary for the inequalities to be obeyed.

CONCLUSION

It is often claimed that a violation of Bell-type inequalities implies that either realism or Einstein locality should be abandoned. As we saw in our counterexample which is both Einstein local and realistic in the common sense of the word, it is only the one-to-one correspondence of the binary variables representing the experimental data with the logical elements of Boole that matters. In general, an inequality cannot be blindly applied to any set of experimental data, a model or a theory [7]. The inequality should be derived in the proper context and conditions and conclusions belonging to the respective derivations cannot simply be mixed.

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